Total Relativistic Energy

Let look at the equation for the relativistic kinetic energy of a moving particle again.

$$KE = \frac{mc^2}{\sqrt{1 - v^2/c^2}} - mc^2$$

→ It separates into two terms:
- 1st depends on speed of the particle
- 2nd is a constant term *independent* of motion

→ KE can be interpreted as the *difference* between a **total energy** term depending on motion and a constant **rest energy** term independent of motion.

$$KE = \underbrace{\frac{mc^2}{\sqrt{1 - v^2/c^2}}}_{\text{total relativistic energy } (E)} - \underbrace{mc^2}_{\text{rest energy}} = E - mc^2$$

Total Relativistic Energy

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}}$$

E is the **total relativistic energy** for a relativistic particle.

and, its **kinetic energy** *KE* is given by,

$$KE = E - mc^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}} - mc^2 = (\gamma - 1)mc^2$$

Total Relativistic Energy

Note 1: for $v = 0$, $\gamma = 1$ and $KE = (\gamma - 1)mc^2 = (1-1)mc^2 = 0$

- $KE = 0$ for a particle at rest is expected…

- But, there is a residual Total Relativistic Energy $E = mc^2 \neq 0$ even for a particle *at rest*. The quantity mc^2 is called the **Rest Energy**.

Rest Energy (mc^2) →

- Independent of velocity
- Proportional to the mass of the particle
- *Mass is a form of energy*

Total Relativistic Energy

Note 2: Since $E = \gamma mc^2$ is the **total** relativistic energy of the system,

→ ***E* is *conserved* in *all* processes !**

- It combines with the two classical independent conservation laws:

  - conservation of energy
  - conservation of mass

- The statement on the Conservation of Total Relativistic Energy is more *general*

Total Relativistic Energy

Note 3:

→The mass "m" which we have been using is a *constant* in our analysis. It is called the **rest mass** ("proper" mass) and is the mass of an object measured by an observer *stationary* with the object.

→The quantity $m_{\text{rel}} = \gamma m$ is called the "relativistic mass" and is *not* a constant for a moving object and is measured by an observer *not* at rest with the object.

Relativistic Momentum & Energy

Relativistic Momentum:

$$\vec{\mathbf{P}} = \gamma m\vec{\mathbf{v}} = \frac{m\vec{\mathbf{v}}}{\sqrt{1 - v^2/c^2}}$$

Momentum of a particle moving with velocity $\vec{\mathbf{v}}$ as measured in the lab frame (S-frame).

Relativistic Energy:

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}}$$

Total energy of a particle moving with velocity $\vec{\mathbf{v}}$ as measured in lab frame (S-frame).

Conservation Laws

The following set of equations form the generalized conservation laws in Special Relativity.

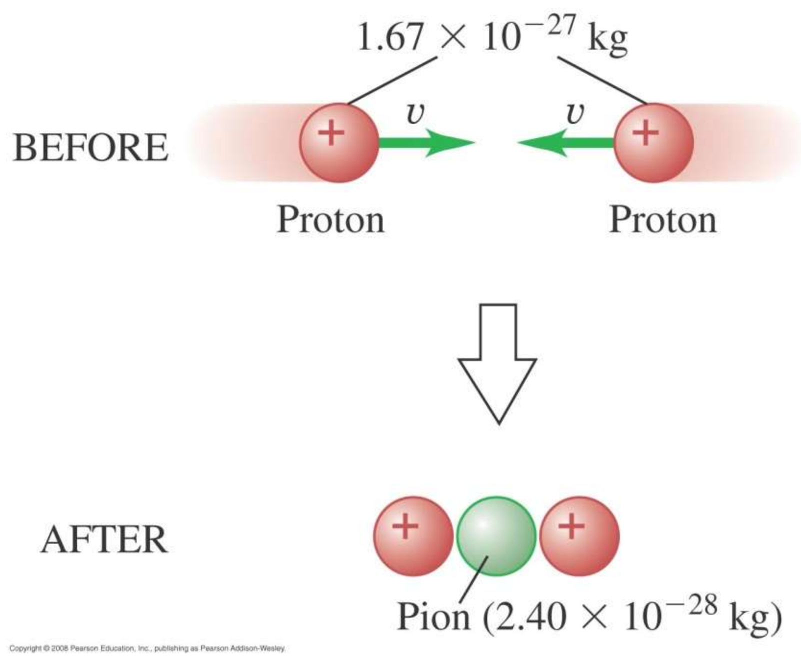
$$\begin{cases} E_i = E_f & \text{Conservation of Relativisity Energy} \\ \vec{\mathbf{P}}_i = \vec{\mathbf{P}}_f & \text{Conservation of Relativistiy Momentum} \end{cases}$$

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}}$$

$$\vec{\mathbf{P}} = \gamma m\vec{\mathbf{v}} = \frac{m\vec{\mathbf{v}}}{\sqrt{1 - v^2/c^2}}$$

AND, these conservation laws apply to *all* processes equally in *all* inertial reference frames !

Example 37.11: A Relativistic Collision



Initially, both protons move in opposite directions, net linear momentum is zero. The three particles after collision are *at rest* again with net linear momentum equals to zero. Thus, relativistic momentum is by design conserved.

We now need to consider the conservation of total relativistic energy.

Question: Find initial velocity of proton.

Example 37.11

Before After

$$2\left(\gamma M_{proton}c^2\right) \quad = \quad 2M_{proton}c^2 + m_{pion}c^2$$

$$2\left(\gamma - 1\right)M_{proton} = m_{pion}$$

$$\gamma = 1 + \frac{m_{pion}}{2M_{proton}} = 1 + \frac{2.40\times10^{-28}\,kg}{2\left(1.67\times10^{-27}\,kg\right)} = 1.072$$

$$\gamma^2 = \frac{c^2}{c^2 - v^2} \quad \rightarrow \quad \gamma^2 c^2 - \gamma^2 v^2 = c^2 \quad \rightarrow \quad v = c\sqrt{1 - 1/\gamma^2}$$

$$v = c\sqrt{1 - 1/1.072^2} = 0.360c$$

Note: Mass *by itself* is not conserved in this process!

New Energy Units

Electron Volt (eV):

The energy equals to moving one positive test charge $e+$ (1 Coulomb) across an electric potential of 1 volt.

$$1eV = (1.6022 \times 10^{-19} C)(1V) = 1.6022 \times 10^{-19} J$$

Example: Rest Mass Energy of an electron $m_e = 9.109 \times 10^{-31} kg$

$$E_0 = m_e c^2 = 9.109 \times 10^{-31} kg \left(2.997 \times 10^8 m/s\right)^2$$

$$= 8.18171 \times 10^{-14} J \left(\frac{1eV}{1.6022 \times 10^{-19} J} \right) = 5.11 \times 10^5 eV = 0.511 MeV$$

$m_e = 0.511 MeV/c^2$ ← (mass of e in units of eV and c)

Energy-Momentum Relation

$$E = \gamma mc^2 \qquad\qquad p = \gamma mv$$

$$E^2 = \gamma^2 m^2 c^4 \qquad\qquad c^2 p^2 = \gamma^2 m^2 v^2 c^2$$

$$E^2 = \frac{m^2 c^4}{1 - v^2/c^2} \qquad\qquad c^2 p^2 = \frac{m^2 v^2 c^2}{1 - v^2/c^2}$$

$$\Rightarrow \quad E^2 - c^2 p^2 = m^2 c^4 \left(\frac{1}{1 - v^2/c^2} - \frac{v^2/c^2}{1 - v^2/c^2} \right)$$

$$E^2 - c^2 p^2 = m^2 c^4$$

Energy-Momentum Relation

$$E^2 - c^2 p^2 = m^2 c^4$$

Similar to the space-time interval $ds^2 = dx^2 - c^2 dt^2$ which is *invariant* for *all* inertial observers (independent of relative motion), the combination $E^2 - c^2 p^2$ is also independent of motion and is an *invariant* quantity

→ Both *E* and *P* will change depending on the relative *S*-*S*' velocity
but $E^2 - c^2 p^2$ will not.

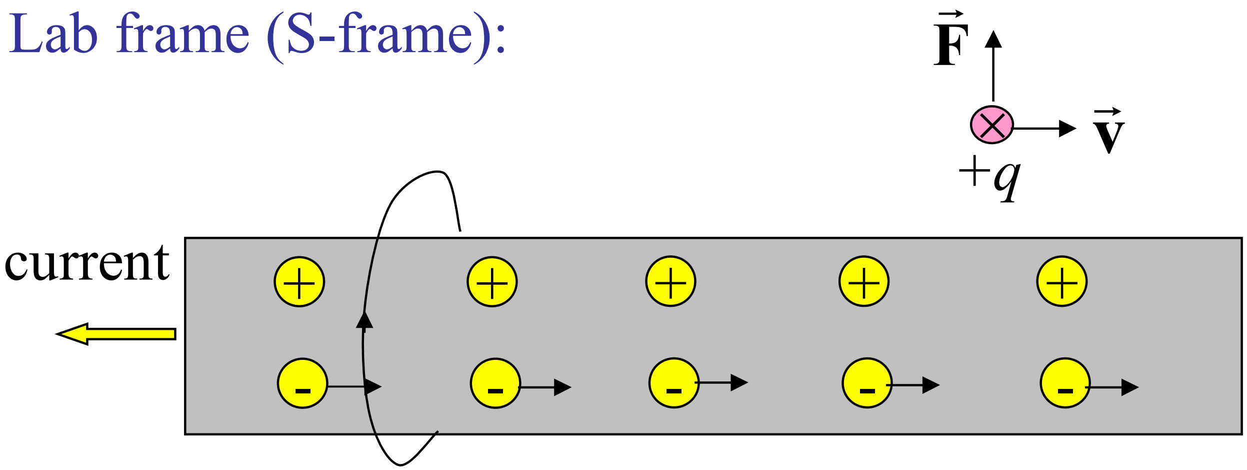
Note:
For particles *at rest*, $p = 0$, this expression gives $E = mc^2$ which is the rest mass energy as previously.

For *photons* with no mass, *E=pc,* which can also be shown from Maxwell's Equations.

Relativity and EM

Consider the force on a moving test charge q from a neutral current carrying wire:

Lab frame (S-frame):



$$\vec{\mathbf{F}} = q\vec{\mathbf{v}} \times \vec{\mathbf{B}}$$

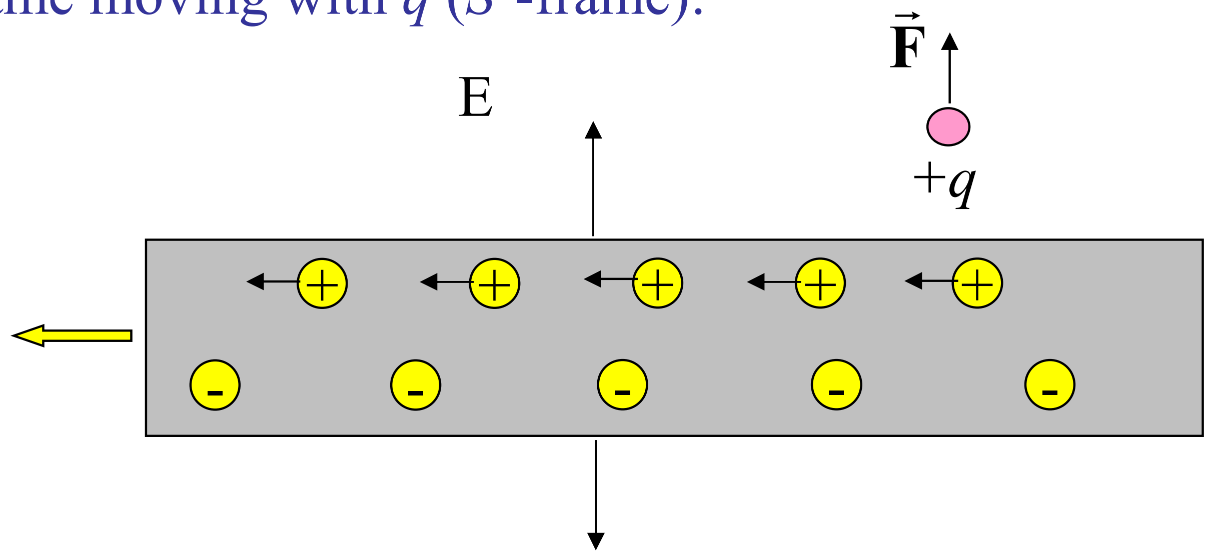
We further assume that the test charge q moves with the same velocity $\vec{\mathbf{v}}$ as the drift velocity of the negative charges in the wire, then $\vec{\mathbf{B}}$ is into the page at q.

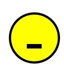
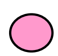
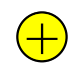
The moving test charge q will feel an upward *magnetic* force.

There is *no* net electric force: wire is assumed to be neutral.

Relativity and EM

Frame moving with q (S'-frame):

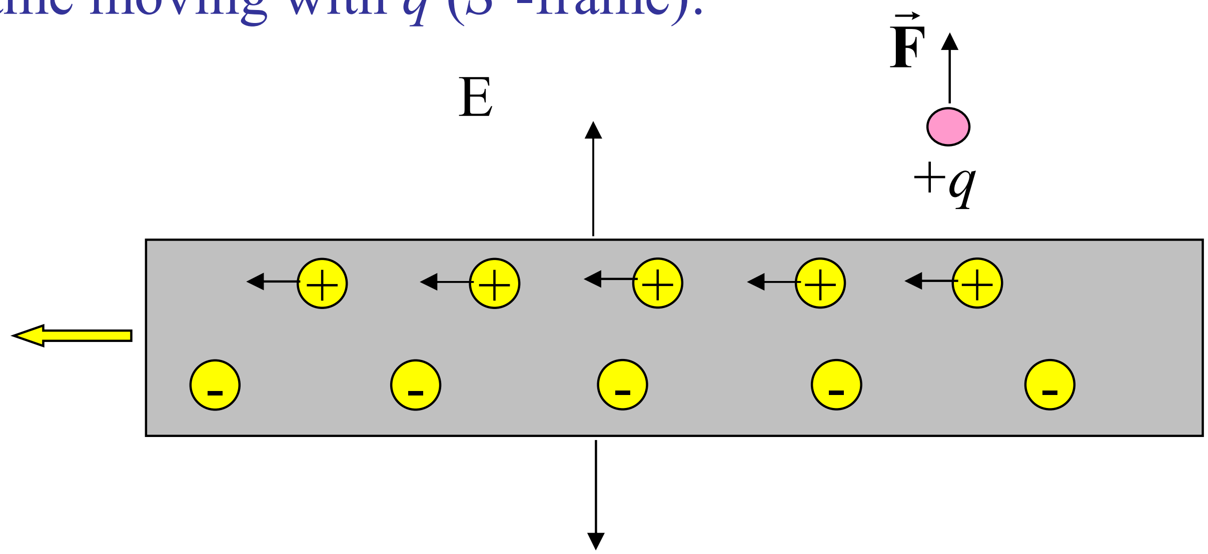


In moving frame, both  and  are stationary but by design  are now moving to the left creating the same current.

$+q$ is not moving so that there is *no* magnetic force due to current.

Relativity and EM

Frame moving with q (S'-frame):



- is further separated since proper distance is larger than measured in the lab frame (S – frame)

+ is tighter together due to length contraction

Now, there is a net charge on the wire in this frame !

q will again feel a force pushing it upward but the force is purely *electric* in this case.

PHYS 262

George Mason University

Prof. Paul So

Chapter 38: Light Waves Behaving as Particles

- Photoelectric Effects
- X-ray Production
- Compton Scattering & Pair Production
- Heisenberg Uncertainty Principle
- Wave/Particle Duality

Quantum Nature of Light

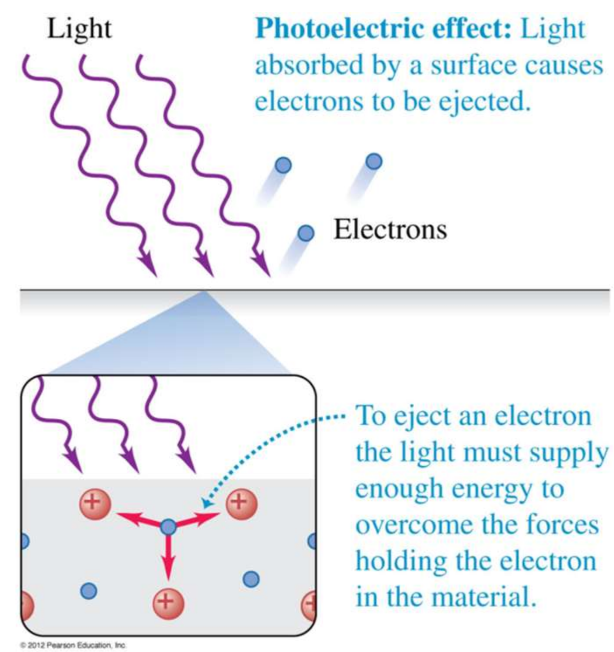
Two seemingly paradoxical nature of light (EM waves):

→ By the end of the 19^{th} century, most physicists (Maxwell, Hertz, and others) have firmly established that electromagnetic waves are waves which exhibit interference and diffraction (Ch. 35-36).

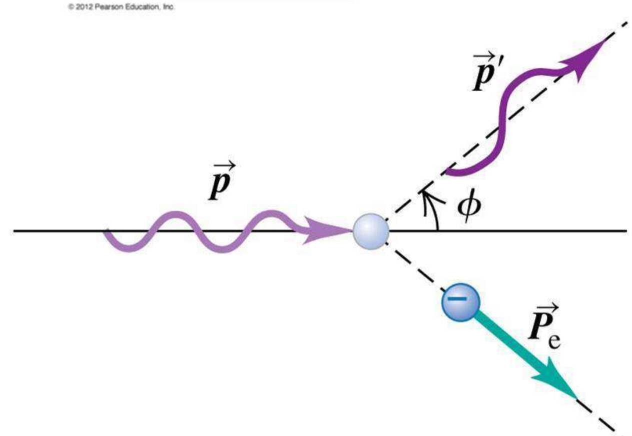
→ But newer experiments on the emissions and absorptions of EM waves have shown behaviors which CANNOT be explained with light being a wave… It requires a radical new thinking of light as *quantized packets of energy* called **photons** (as particles).

Quantum Nature of Light

Photoelectric Effect: When light struck a metal surface, some electrons near the surface will be emitted. The absorption and emission process can only be explained by assuming light is *quantized* into packets of energy.



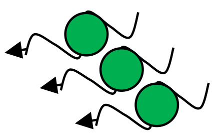
X Rays Production, Compton Scattering, & Pair Production:
X-rays were discovered in 1895 in high-voltage electric discharge tubes but no one understood the process in their production and what determine their wavelengths. In particular, when x-rays collide with matter, the scatter rays act non-classically.



Light as Particles (Photons)

In 1905, Einstein published his theory on photoelectric effect which resulted in his Nobel prize in 1921.

Built upon Max Planck's hypothesis of quantized light (photon). [later]



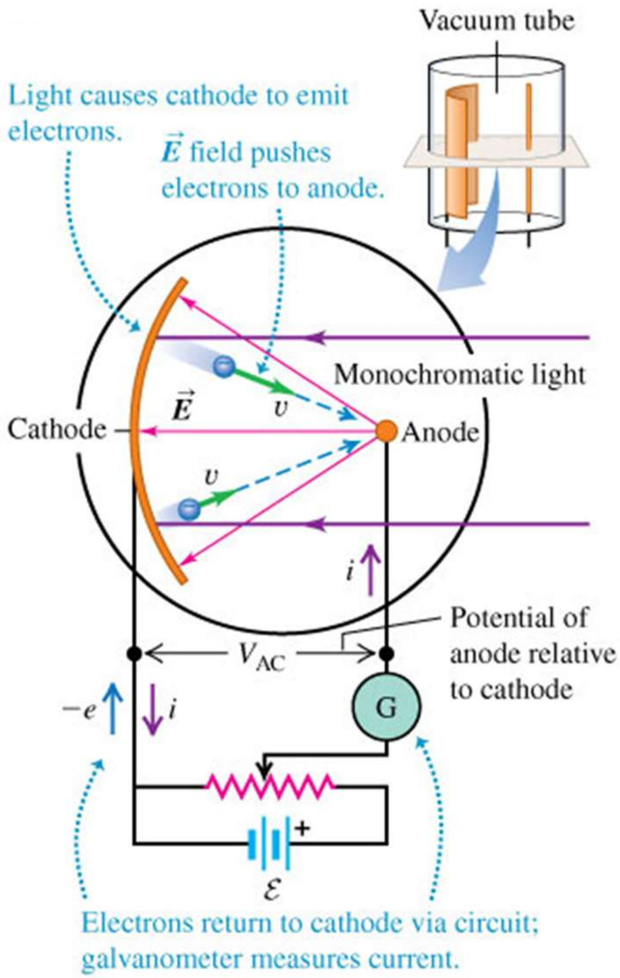
$$E = hf = \frac{hc}{\lambda}$$ (energy of a photon)

where $h = 6.626 \times 10^{-34}\, J \cdot s$ is a universal constant called **Planck's Constant**.

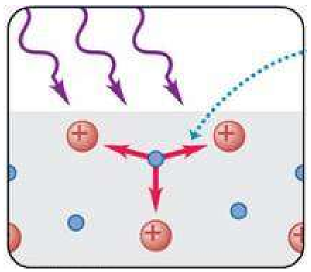
Note the smallness of this number.

The Photoelectric Effect

An experimental demonstration of the *particle* nature of light.



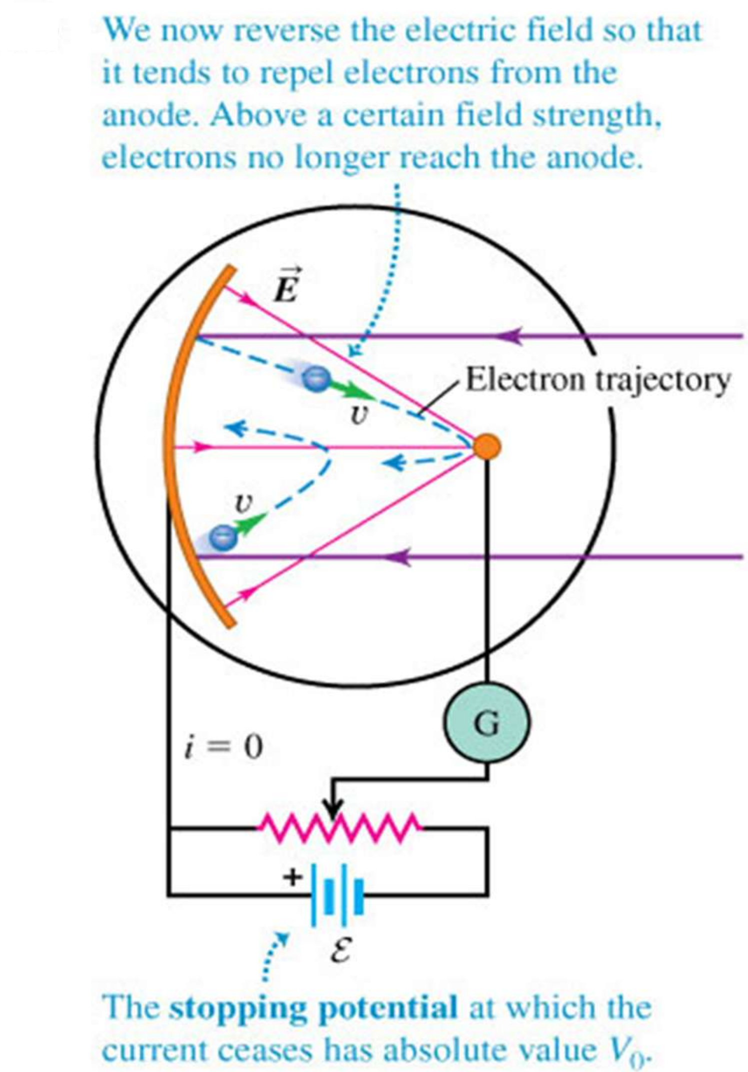
Electrons on the metal surface (cathode) are normally bounded to the positive ions on the surface.



The potential energy which an e needs to escape the surface is call the **work function** ϕ

V_A can help (+) or inhibit (-) the ejected e to get from cathode to anode.

The Photoelectric Effect



When $V_{AC}(\text{reversed}) \rightarrow -V_0$

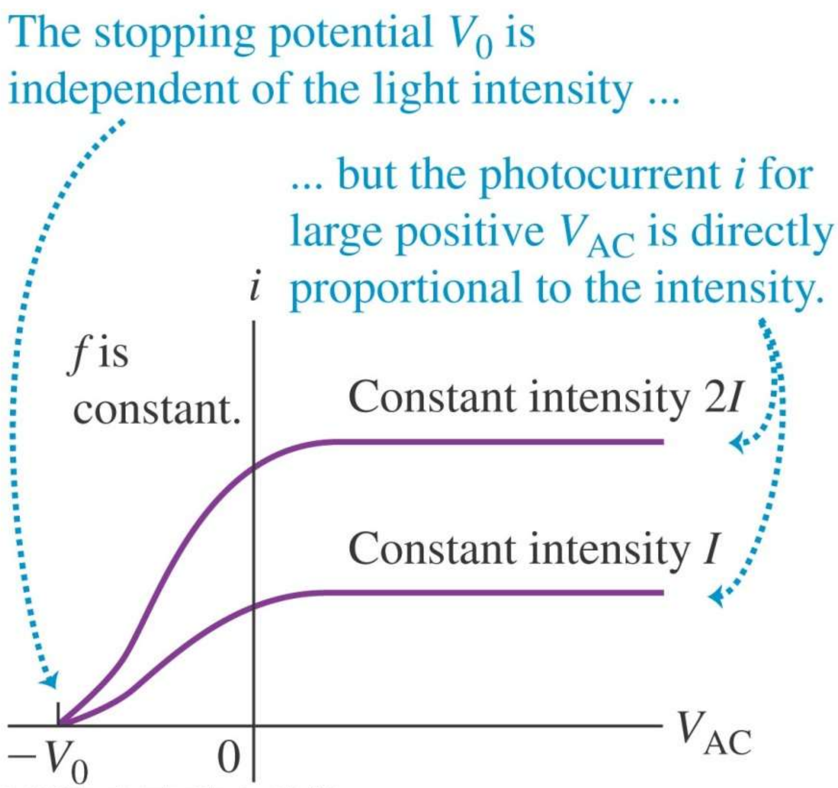
Above a certain potential strength V_0 , NO e- can reach the anode!

The minimum V_0 needed to stop all e- getting across to the anode is called the **stopping potential** V_0 and

V_0 is basically a direct measurement of the maximum KE (K_{max}) of these electrons and they are related by,

$$K_{max} = eV_0$$

The Photoelectric Effect



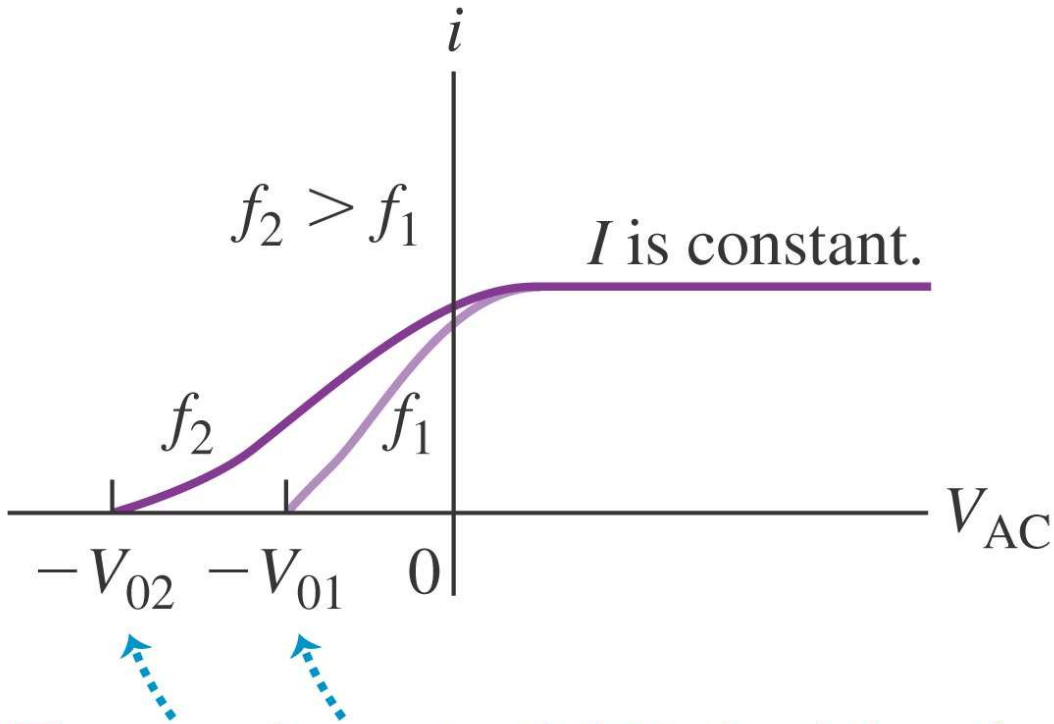
Classical Expectation:

- Energy of EM wave depends on intensity → emission will monotonically depend on intensity
- Energy of light not dependent on f
- For low intensity light, emission is expected to be delayed

Unexpected Results:

- *No* electrons are ejected if $f < f_0$ (**threshold frequency**) independent of light intensity
- Even with $f > f_0$ but at very low intensity emission is *immediate* (not delayed)
- V_0 is *independent* of intensity

Photoelectric Effect



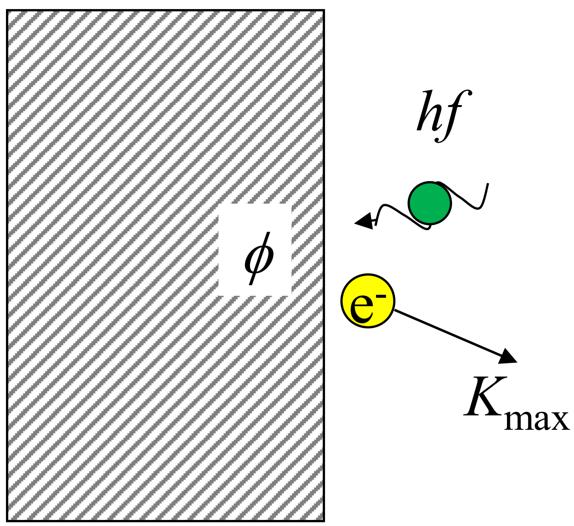
The stopping potential V_0 (and therefore the maximum kinetic energy of the photoelectrons) increases linearly with frequency: since $f_2 > f_1$, $V_{02} > V_{01}$.

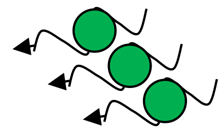
More Unexpected Result:

The dependence of V_0 (K_{max} of the ejected electrons) on f is also another unexpected (unexplainable by classical physics) result.

→ Energy of light was NOT expected (classically) to depend on its frequency f.

Einstein's Photon Explanation





Incident light as a collection of photons (particles)

- Each photon has energy hf
- Intensity of light ~ # of photons

The interaction is an *all-or-none* process. **Electrons bounded to the surface of the metal can absorb a single photon at a time or none at all.** If hf is large enough to overcome ϕ, an electron will be ejected with kinetic energy $K_{\max}$.

By energy conservation, we have: $K_{\max} = hf - \phi$ — ϕ depends on the metal surface

$$\Longrightarrow \quad eV_0 = hf - \phi$$

Einstein's Photon Explanation

Unexpected Results Explained:

- Since $K_{\max}$ has to be positive, if $hf < \phi$, no electrons will gain enough kinetic energy to leave. So, there has to be a threshold frequency f_o and is given by,

$$hf_0 = \phi$$

- Since intensity I is only proportional to the # of photons and the interaction is for individual photons, $K_{\max}$ (or V_0) depends only on the energy of each *individual* photon hf and not on the light's intensity, i.e.,

→ Increasing intensity will only increase the # of photo-electrons being ejected and it will increase the photocurrent being observed but it will NOT affect the stopping potential V_0.

- More importantly, $K_{\max}$ (V_o) linear dependence on f is explicit with the equation

$$K_{\max} = eV_0 = hf - \phi$$

Notes

- Convenient Energy Units:

1eV = energy required to move one unit of charge across an electric potential of 1 V.

$$1\text{eV} = \left(1.602\times10^{-19}\,C\right)\left(1V\right) = 1.602\times10^{-19}\,J$$

$$h = 6.626\times10^{-34}\,J\cdot s\left(\frac{1eV}{1.602\times10^{-19}\,J}\right) = 4.136\times10^{-15}\,eV\cdot s$$

$$hc = 4.136\times10^{-15}\,eV\cdot s\left(3.00\times10^{8}\,m/s\right) = 1.241\times10^{-6}\,eV\cdot m = 1241eV\cdot nm$$

- Energy and Momentum of a Photon:

$$P = \frac{E}{c} = \frac{hf}{c} = \frac{h}{\lambda} \qquad \text{(momentum of a photon)}$$

- The duality of light (wave & particle) applied to the entire EM spectrum !

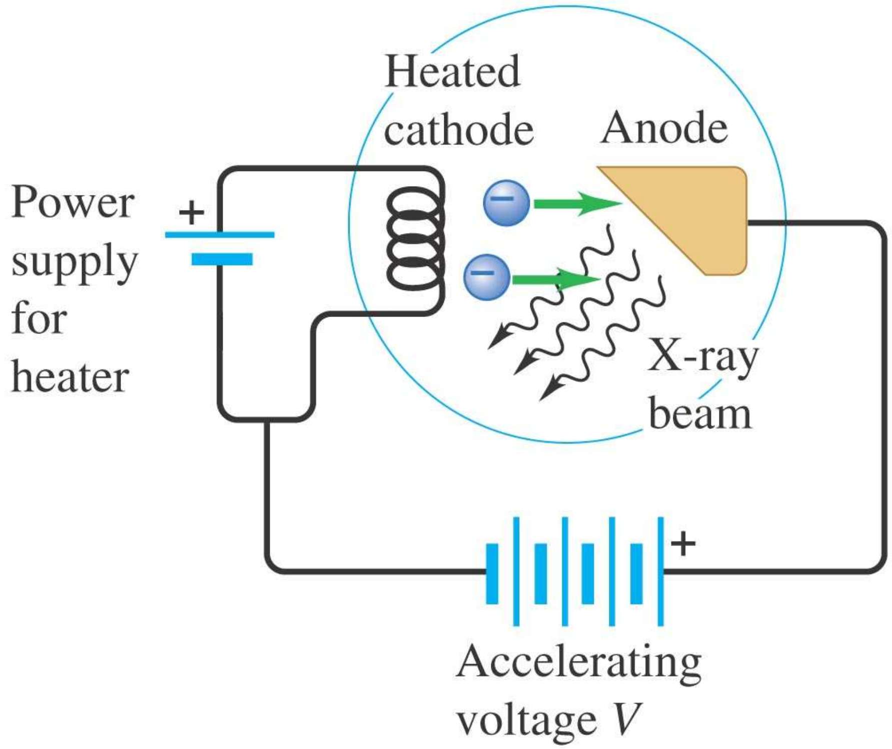
Emission of Light as Particles: X-Ray Production

X-rays are produced when rapidly moving electrons that have been accelerated through a *large* potential difference (10^3 to 10^6 V) strike a metal target.

X-rays emission is the *inverse* of the photoelectric effect.

Photoelectric: $hf \rightarrow K$ of e
X-Ray prod: K of $e \rightarrow hf$

Electrons are emitted thermionically from the heated cathode and are accelerated toward the anode; when they strike it, x rays are produced.



Discrete Set of λs :Two Processes

Classical EM → *deceleration* of high energy e^- should produce EM waves in a *broad* range of f.

BUT, the observation is that …

1. Independent of target material: *bremsstrahlung* (braking radiation) → gives maximum f (energy) or minimum λ directly proportional to the energy of the accelerated eletrons.

$$eV_{AC} = hf_{\max} = \frac{hc}{\lambda_{\min}}$$

max *KE* of accelerated *e*

2. Dependent on target material, a *characteristic* spectrum of X-Rays will also be emitted → electrons with sufficient *KE* can excite atoms in the target material. When they decay back to their ground state, light (X-Rays) will be emitted.

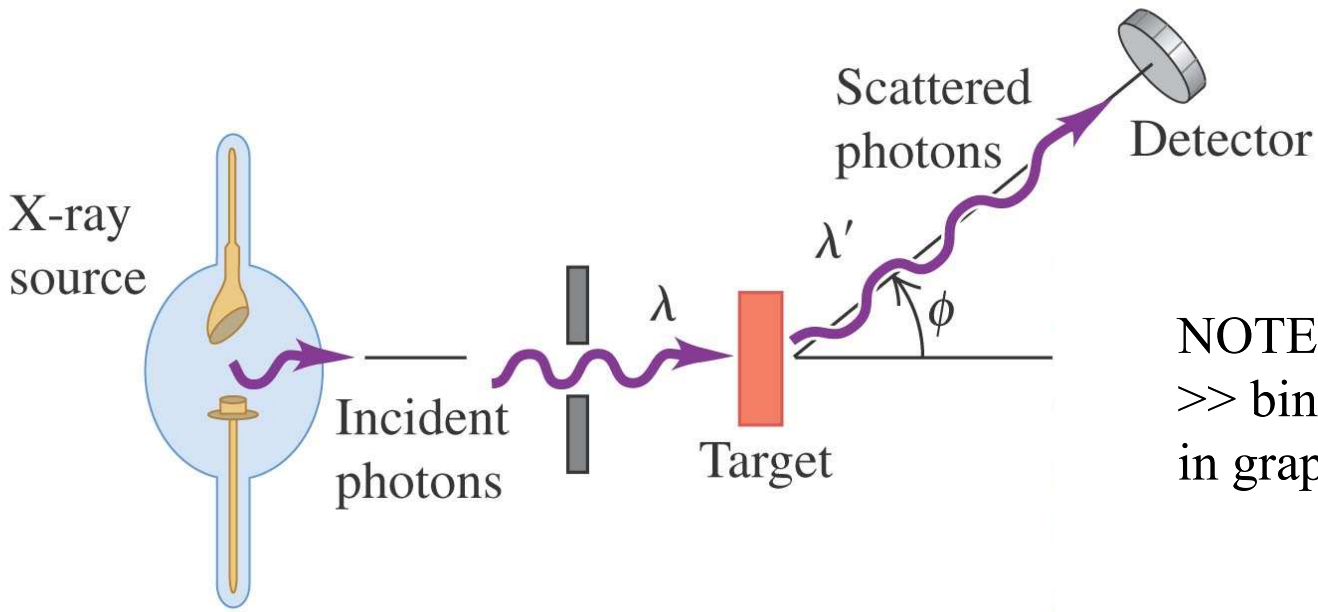
Medical Applications of X-Rays

High-energy photons (such as X-Rays) can penetrate denser materials such as bones which low-energy photons (such as visible light) can not. Then, by measuring the degree of penetration, one can map out different biology structures in your body.

High-energy photons can also damage biological tissues by breaking molecular bonds and creating highly reactive free radical such as H or OH .

Compton Scattering

In 1923, Arthur H. Compton provided an additional direct conformation on the quantum nature of x-rays.



NOTE: Energy of photon >> binding energy of e^- in graphite.

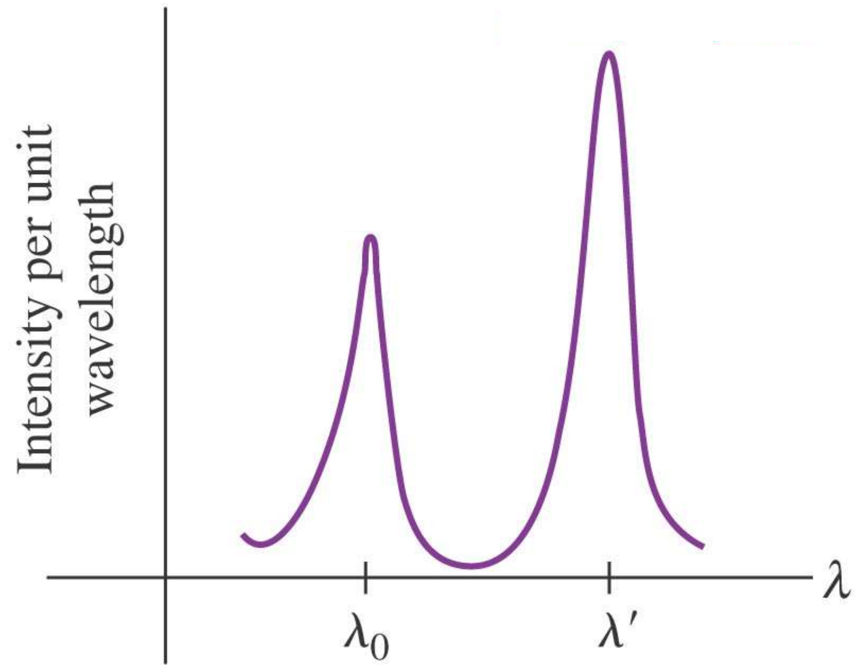
- X-rays of well-defined λ are made to fall on a graphite target
- For various scattering angle ϕ, intensities of scattered x-rays are measured as a function of the wavelength.

Compton Scattering

The scattered x-rays have intensity peaks at *two* wavelengths: λ_0 and λ'.

$$\Delta\lambda = \lambda' - \lambda_0$$

is called the *Compton Shift*.



Compton Scattering

Classical prediction:

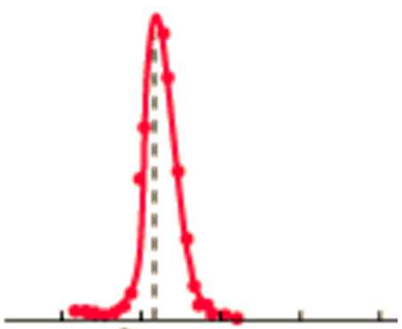
Electrons in graphite absorbs x-rays and reemit them back.

- If the electron is stationary, then the reemitted $\lambda' = \lambda_0$.
- But electrons are moving before & after scattering, so that λ' will be Dopper's shifted depending on e^-'s velocity.

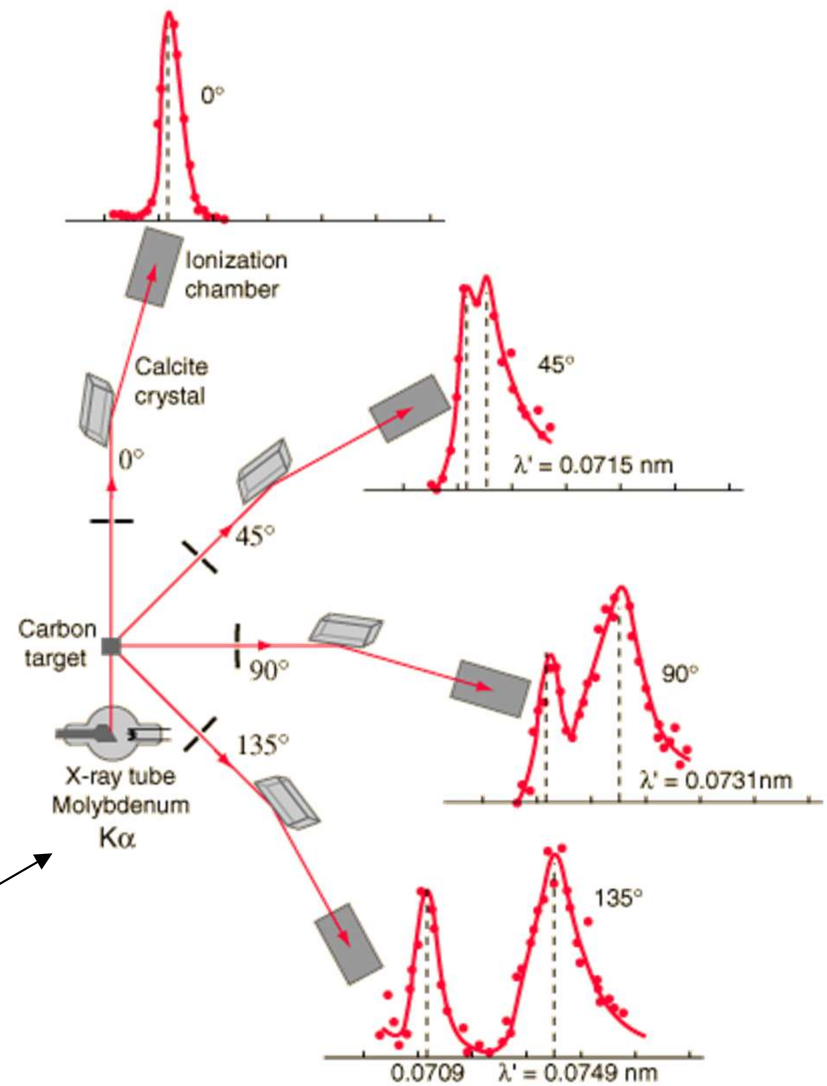
Compton Scattering

Classical prediction:

Since diff. electrons will have diff. velocity, the intensity profile for the scattered x-rays is expected to be a single peak with a spread around λ_0.



But, the actual experiment gives *two peaks* with a *Compton Shift* which depends on ϕ. Classical physics can't explain this !



Compton Scattering

Quantum Explanation:

Compton and his co-workers showed that if we take the x-rays as particles with

$$E = hf$$

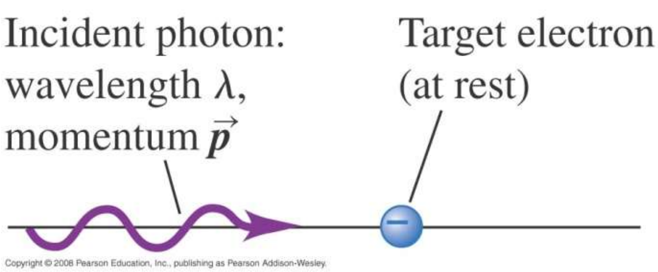
$$|p| = hf / c$$

and treat the scattering process as a "billiard-like" collision,

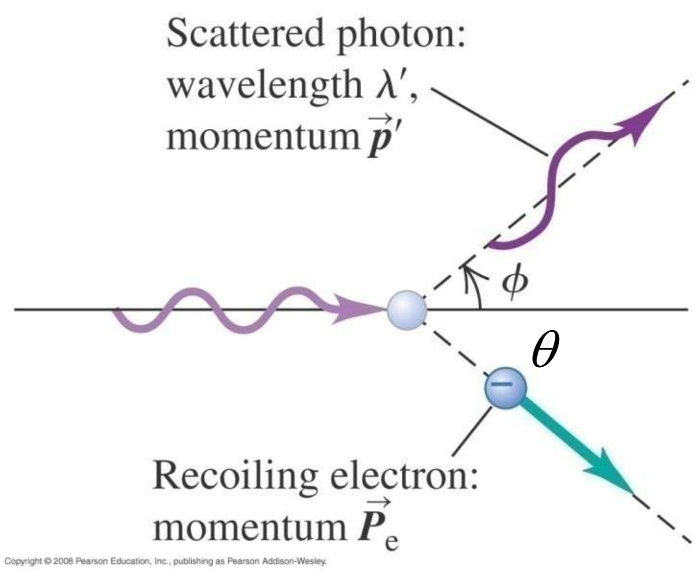
then, the observation of $\Delta\lambda$'s dependence on ϕ can be explained !

Compton Scattering

(a) Before collision: The target electron is at rest.



(b) After collision: The angle between the directions of the scattered photon and the incident photon is ϕ.



Recall:

$$E_{photon} = hf = \frac{hc}{\lambda}$$

$$\left|\vec{\mathbf{P}}_{photon}\right| = \frac{h}{\lambda}$$

$$E_{electron} = \gamma mc^2$$

$$\vec{\mathbf{P}}_{electron} = \gamma m\vec{\mathbf{v}}$$

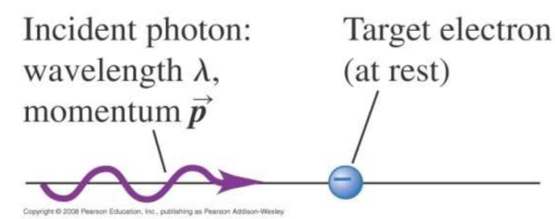
Compton Scattering

Conservation of Total Relativistic Energy:

Note: high energy collision → e's speed might be relativistic !

(before) (after)

$$\frac{hc}{\lambda} + mc^2 = \frac{hc}{\lambda'} + \gamma mc^2$$

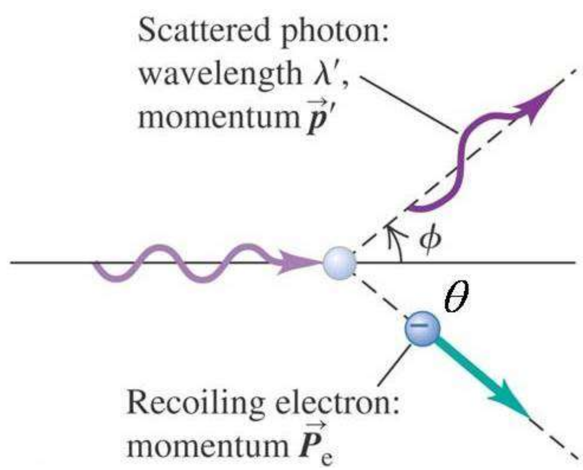


Conservation of Relativistic Momentum:

(before) (after)

$$x-dir: \quad \frac{h}{\lambda} = \frac{h}{\lambda'}\cos\phi + \gamma mv\cos\theta$$

$$y-dir: \quad 0 = \frac{h}{\lambda'}\sin\phi - \gamma mv\sin\theta$$



(knowns → λ: given, ϕ: observation angle) 3 eqs. with 3 unknowns (θ, λ', v) !