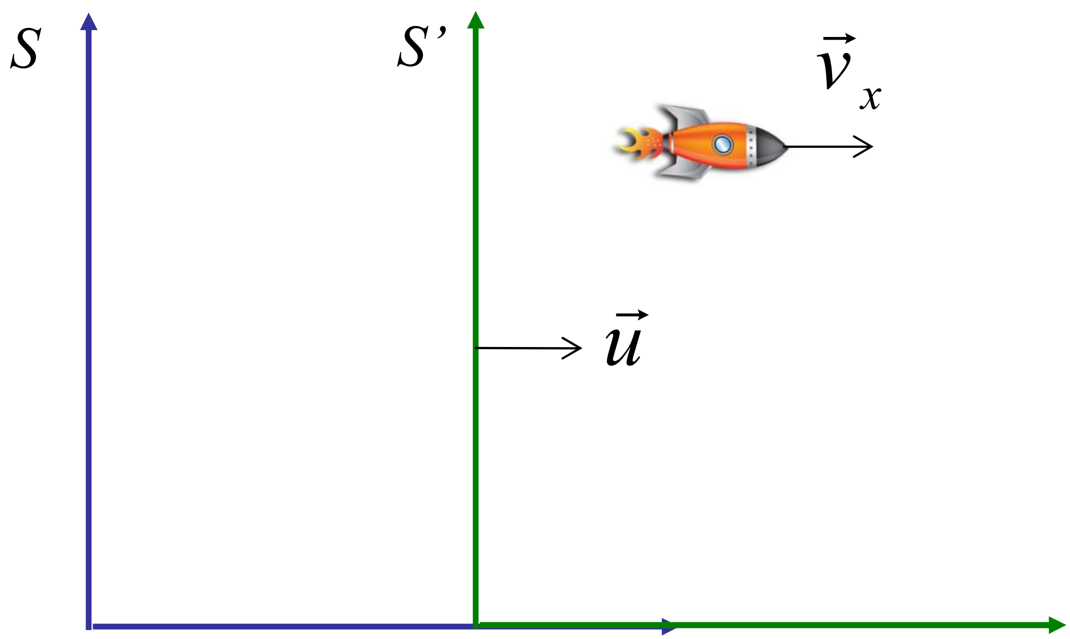
Lorentz Velocity Transformation

In S- frame, let say that we have an object moving in the x-direction with speed,

$$v_x = \frac{dx}{dt}$$



Note:
u → relative speed between S and S'
v_x → speed of object

Lorentz Velocity Transformation

Lorentz Transform gives: $$\begin{cases} dx = \gamma\left(dx' + u\,dt'\right) \\ dt = \gamma\left(dt' + \dfrac{u}{c^2}dx'\right) \end{cases}$$

$$\Rightarrow \quad v_x = \frac{dx}{dt} = \frac{\gamma\left(dx' + u\,dt'\right)}{\gamma\left(dt' + \dfrac{u}{c^2}dx'\right)}$$

$$v_x = \frac{\cancel{\gamma}}{\cancel{\gamma}}\,\frac{dx' + u\,dt'}{dt' + \dfrac{u}{c^2}dx'}\left(\frac{1/dt'}{1/dt'}\right) = \frac{\dfrac{dx'}{dt'} + u}{1 + \dfrac{u}{c^2}\dfrac{dx'}{dt'}}$$

Lorentz Velocity Transformation

In S'- frame, the velocity of the object is defined as,

$$v'_x = \frac{dx'}{dt'}$$

This then gives,

$$v_x = \frac{\frac{dx'}{dt'} + u}{1 + \frac{u}{c^2}\frac{dx'}{dt'}} \Rightarrow \boxed{v_x = \frac{v'_x + u}{1 + \frac{u}{c^2} v'_x}}$$

(Lorentz Velocity Transformation)

Slow relative speed ($u << c$):

$$u / c << 1$$

$$\Rightarrow \quad v_x \cong \frac{v'_x + u}{1 + 0} = v'_x + u$$

(Galilean Velocity Transform)

Object moving at the speed of light:

$$v'_x = c$$

$$\Rightarrow \quad v_x = \frac{c + u}{1 + \frac{u}{c^{\cancel{2}}}\cancel{c}} = \frac{c(1 + u/c)}{1 + u/c} = c$$

(c is the same in all frames)

Lorentz Velocity Transformation

If the object has velocity components in *y* and *z* directions: v'_y & v'_z, how would these components transform (u is in x direction only)?

$$v_y = \frac{dy}{dt} = \frac{dy'}{\gamma\left(dt' + \frac{u}{c^2}dx'\right)}\left(\frac{1/dt'}{1/dt'}\right) = \frac{\frac{dy'}{dt'}}{\gamma\left(1 + \frac{u}{c^2}\frac{dx'}{dt'}\right)}$$

$$v_y = \frac{v'_y}{\gamma\left(1 + \frac{u}{c^2}v'_x\right)}$$

Similarly for the *z*-component !

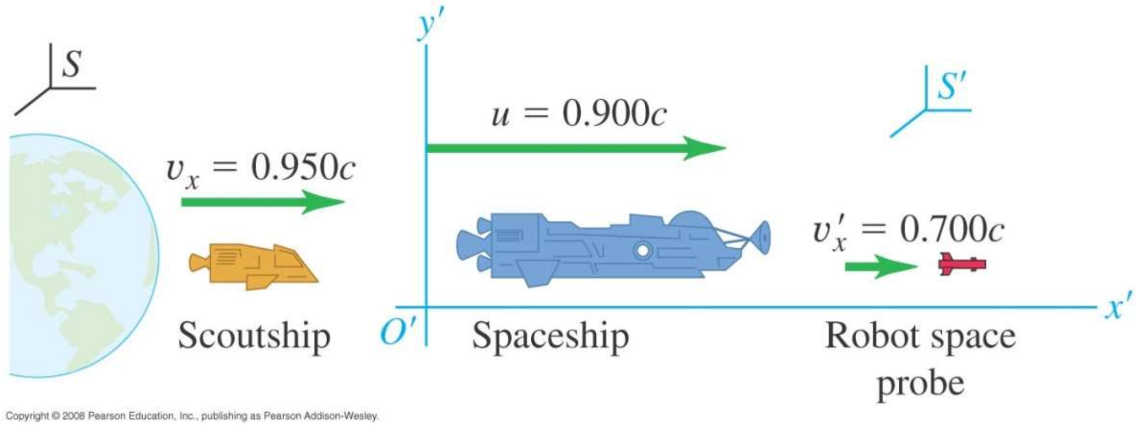
Lorentz Velocity Transform

From the principle of relativity, there should be no physical distinction for the two inertial observers in relative motion.

So the Lorentz Velocity Transform equation and its inverse transform should have the same form but with $u \leftrightarrow -u$ for the inverse transform of ***v*** in term of ***v'***.

$$\begin{cases} v_x = \dfrac{v'_x + u}{1 + \dfrac{u}{c^2} v'_x} \\ v_y = \dfrac{v'_y}{\gamma\left(1 + \dfrac{u}{c^2} v'_x\right)} \end{cases} \qquad \begin{cases} v'_x = \dfrac{v_x - u}{1 - \dfrac{u}{c^2} v_x} \\ v'_y = \dfrac{v_y}{\gamma\left(1 - \dfrac{u}{c^2} v_x\right)} \end{cases}$$

Example 37.7



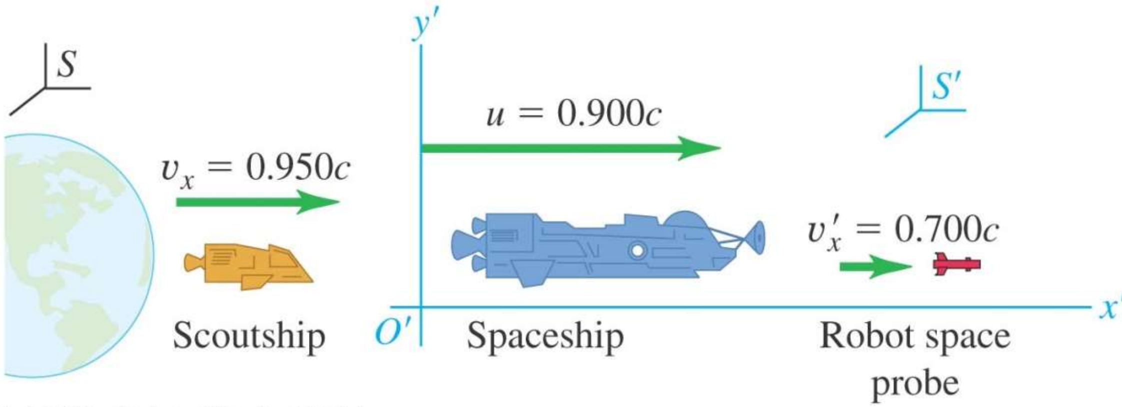
a) What is the probe's velocity relative to earth?
b) What is the scoutship's velocity relative to spaceship?

Setup: Two frames: S→Earth, S'→ Spaceship, $u = +\,0.900c$

a) In S'-frame, the probe moves at $v_x'(probe) = 0.700c$

$$v_x = \frac{v'_x + u}{1 + uv'_x / c^2} = \frac{0.700c + 0.900c}{1 + (0.700)(0.900)} = 0.982c$$

Example 37.7



a) The scoutship's speed is given in S-frame (with respect to Earth),

$$v_x\left(scout\right) = +0.950c$$

The scoutship's speed with respect *S*'-frame is given by the *inverse* transform,

$$v'_x = \frac{v_x - u}{1 - uv_x / c^2} = \frac{0.950c - 0.900c}{1 - (0.950)(0.900)} = 0.345c$$

Relativistic Momentum & Energy

As we have seen, time intervals, length intervals, and velocity change according to the Lorentz Transformation depending on the observer's frame of reference.

$$(x,t) \quad \leftrightarrow \quad (x',t') \quad \text{Lorentz Transformation}$$

Other *dynamical quantities* (such as momentum, energy, etc.) must also be appropriately expressed so that the laws of physics satisfy the following conditions:

- Satisfy the two postulates of Special Relativity:
    - Laws of physics (e.g., conservation of momentum, conservation of energy, Newton's laws) apply equally to all inertial observers.
    - Speed of light in vacuum same for all inertial observers
- The modified relativistic dynamical quantities should reduce to the classical ones for $u << c$.

Relativistic Momentum & Energy

Relativistic Momentum:

$$\vec{\mathbf{P}} = \gamma m \vec{\mathbf{v}} = \frac{m\vec{\mathbf{v}}}{\sqrt{1 - v^2/c^2}}$$

Momentum of a particle moving with velocity $\vec{\mathbf{v}}$ as measured in the lab frame (S-frame).

Relativistic Energy:

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}}$$

Total energy of a particle moving with velocity $\vec{\mathbf{v}}$ as measured in lab frame (S-frame).

Relativistic Momentum & Energy

All laws of physics remain valid in all inertial reference frames means.

→ Conservation Laws for relativistic $\vec{\mathbf{P}}$ & E must remain the same!

→ Experimentally, it has been shown repeatedly that it is ($\gamma m\vec{\mathbf{v}}$ and γmc^2) rather than their classical counterparts that are conserved in high energy collisions!

→ For $v << c$, $\gamma = \dfrac{1}{\sqrt{1-v^2/c^2}} \cong 1$ So, $\vec{\mathbf{P}} = \gamma m\vec{\mathbf{v}} \rightarrow m\vec{\mathbf{v}}$

So that for non-relativistic speeds, relativistic $\vec{\mathbf{P}}$ reduces to classical $\vec{\mathbf{P}}$.

Relativistic Force

$$\vec{\mathbf{F}} = \frac{d\vec{\mathbf{P}}}{dt}$$

Newton's 2nd Law: same form as in the classical case but with relativistic momentum $\vec{\mathbf{P}}$.

Note: If $\vec{\mathbf{F}} = 0$ (no external force), momentum $\vec{\mathbf{P}}$ as expected will be *conserved* in *both* relativistic & classical regimes !

Let use the relativistic force to consider the **work-energy theorem**:

Work done by F to accelerate a relativistic particle from v_i to v_f $\Rightarrow$

$$W = \int_i^f F\,dx = \int_i^f \frac{dP}{dt}\,dx$$

(for simplicity, F is in x-direction only)

Relativistic Work & Energy

$$\int_i^f \frac{dP}{dt}\,dx = \int_i^f \frac{dP}{\cancel{dt}}\,v\,\cancel{dt} \quad \Rightarrow \quad W = \int_i^f v\,dP$$

Now, $d(Pv) = vdP + Pdv \quad \rightarrow \quad vdP = d(Pv) - Pdv$

Substituting this into the above equation for *W*:

$$W = \int_i^f v\,dP = \underbrace{\int_i^f d(Pv)}_{1^{st}\text{ term}} - \underbrace{\int_i^f P\,dv}_{2^{nd}\text{ term}}$$

Relativistic Work & Energy

$1st$ term $= Pv\Big|_i^f = (\gamma m v_f)v_f - (\gamma m v_i)v_i$

For simplicity, choose $v_i = 0$ and $v_f = v$

Then, the 1st term becomes: $\int_i^f d(Pv) = \gamma m v^2 = \dfrac{mv^2}{\sqrt{1-v^2/c^2}}$

Now, let consider the 2nd term,

$$\int_i^f P\,dv = \int_i^f \frac{mv}{\sqrt{1-v^2/c^2}}\,dv$$

We can integrate this by a simple change of variable.

Relativistic Work & Energy

Let $$s = \left(1 - v^2/c^2\right)^{1/2}$$

$$ds = \frac{1}{2}\left(1 - v^2/c^2\right)^{-1/2}\left(-2v/c^2\right)dv$$

$$ds = -\frac{v/c^2}{\sqrt{1 - v^2/c^2}}\,dv$$

So, $$\int_i^f P\,dv = \int_i^f \frac{mv}{\sqrt{1 - v^2/c^2}}\,dv = -mc^2 \int_i^f \frac{-v/c^2}{\sqrt{1 - v^2/c^2}}\,dv = -mc^2 \int_i^f ds$$

$$= -mc^2\left(s_f - s_i\right)$$

$$= -mc^2\sqrt{1 - v^2/c^2} + mc^2\sqrt{1 - 0^2/c^2}$$

$$= -mc^2\sqrt{1 - v^2/c^2} + mc^2$$

Relativistic Work & Energy

Putting everything together,

$$W = \frac{mv^2}{\sqrt{1-v^2/c^2}} + mc^2\sqrt{1-v^2/c^2} - mc^2$$

$$= \frac{mv^2}{\sqrt{1-v^2/c^2}} + mc^2\sqrt{1-v^2/c^2}\,\frac{\sqrt{1-v^2/c^2}}{\sqrt{1-v^2/c^2}} - mc^2$$

$$= \frac{mv^2}{\sqrt{1-v^2/c^2}} + \frac{mc^2}{\sqrt{1-v^2/c^2}}\left(1-v^2/c^2\right) - mc^2$$

$$= \frac{\cancel{mv^2} + mc^2 \cancel{- mv^2}}{\sqrt{1-v^2/c^2}} - mc^2 = \frac{mc^2}{\sqrt{1-v^2/c^2}} - mc^2$$

Relativistic Work & Energy

Applying the **work-energy theorem**, this amount of work done to accelerate the particle from 0 to v should equal to the change in KE.

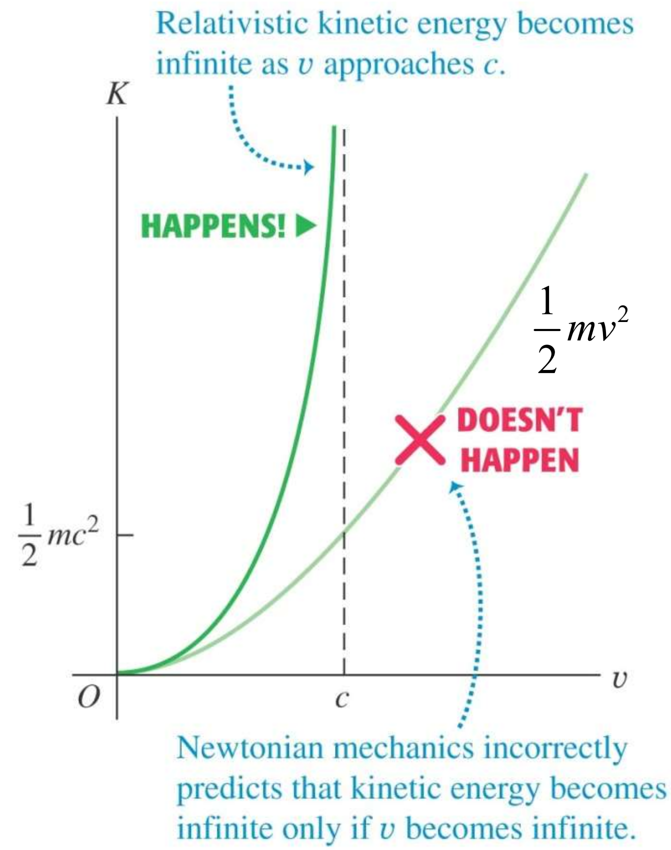
(Laws of physics should be unchanged in relativity !)

Since $v_i = 0, \quad KE_i = 0$

$v_f = v, \quad KE_f = KE$

$$W = \boxed{KE = \frac{mc^2}{\sqrt{1 - v^2/c^2}} - mc^2 = \gamma mc^2 - mc^2}$$

Relativistic Kinetic Energy of a particle moving with $\vec{\mathbf{v}}$ as measured in the lab frame.



Relativistic KE → Classical KE

Slow moving particle regime $v << c$:

Using binomial theorem, $\left(1-\frac{v^2}{c^2}\right)^{-1/2} = 1+\left(-\frac{1}{2}\right)\left(-\frac{v^2}{c^2}\right)+O\left(\left(\frac{v^2}{c^2}\right)^2\right) \cong 1+\frac{1}{2}\frac{v^2}{c^2}$

Substituting this into the equation for Relativistic KE,

$$KE = \left(1-\frac{v^2}{c^2}\right)^{-1/2} mc^2 - mc^2 = \left[\left(1-\frac{v^2}{c^2}\right)^{-1/2} - 1\right]mc^2$$

$$\cong \left[1+\frac{1}{2}\frac{v^2}{c^2}-1\right]mc^2$$

$$= \frac{1}{2}mv^2$$

This is the classical result for $v << c$.

Total Relativistic Energy

Let look at the equation for the relativistic kinetic energy of a moving particle again.

$$KE = \frac{mc^2}{\sqrt{1 - v^2/c^2}} - mc^2$$

→ It separates into two terms:

- 1st depends on speed of the particle
- 2nd is a constant term *independent* of motion

→ KE can be interpreted as the *difference* between a **total energy** term depending on motion and a constant **rest energy** term independent of motion.

$$KE = \underbrace{\frac{mc^2}{\sqrt{1 - v^2/c^2}}}_{\text{total energy}} - \underbrace{mc^2}_{\text{rest energy}} = E - mc^2$$

Total Relativistic Energy

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}}$$

E is the **total relativistic energy** for a relativistic particle.

and, its **kinetic energy** *KE* is given by,

$$KE = E - mc^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}} - mc^2 = (\gamma - 1)mc^2$$

Total Relativistic Energy

Note 1: for $v = 0$, $\gamma = 1$ and $KE = (\gamma - 1)mc^2 = (1-1)mc^2 = 0$

- $KE = 0$ for a particle at rest is expected…

- But, there is a residual Total Relativistic Energy $E = mc^2 \neq 0$ even for a particle *at rest*. The quantity mc^2 is called the **Rest Energy**.

Rest Energy (mc^2) →

- Independent of velocity
- Proportional to the mass of the particle
- *Mass is a form of energy*

Total Relativistic Energy

Note 2: Since $E = \gamma mc^2$ is the **total** relativistic energy of the system,

$\Rightarrow$ ***E* is *conserved* in *all* processes !**

- It combines with the two classical independent conservation laws:

  - conservation of energy
  - conservation of mass

- The statement on the Conservation of Total Relativistic Energy is more *general*

Total Relativistic Energy

Note 3:

→The mass "*m*" which we have been using is a *constant* in our analysis. It is called the **rest mass** ("proper" mass) and is the mass of an object measured by an observer *stationary* with the object.

→The quantity $m_{\text{rel}} = \gamma m$ is called the "relativistic mass" and is *not* a constant for a moving object and is measured by an observer *not* at rest with the object.

Relativistic Momentum & Energy

Relativistic Momentum:

$$\vec{\mathbf{P}} = \gamma m\vec{\mathbf{v}} = \frac{m\vec{\mathbf{v}}}{\sqrt{1 - v^2/c^2}}$$

Momentum of a particle moving with velocity $\vec{\mathbf{v}}$ as measured in the lab frame (S-frame).

Relativistic Energy:

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}}$$

Total energy of a particle moving with velocity $\vec{\mathbf{v}}$ as measured in lab frame (S-frame).

Conservation Laws

The following set of equations form the generalized conservation laws in Special Relativity.

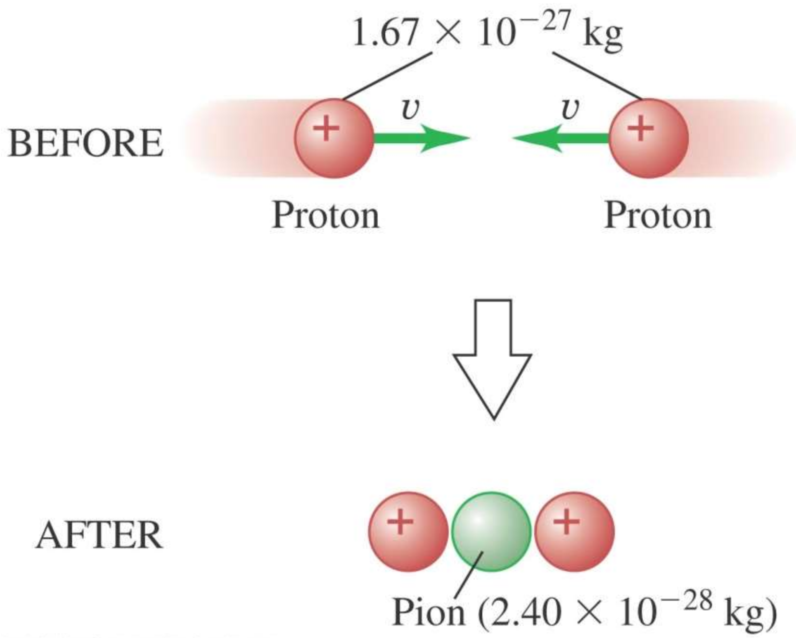
$$\begin{cases} E_i = E_f & \text{Relativisity Energy} \\ \vec{\mathbf{P}}_i = \vec{\mathbf{P}}_f & \text{Relativistiy Momentum} \end{cases}$$

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}}$$

$$\vec{\mathbf{P}} = \gamma m\vec{\mathbf{v}} = \frac{m\vec{\mathbf{v}}}{\sqrt{1 - v^2/c^2}}$$

AND, these conservation laws apply to *all* processes equally in *all* inertial reference frames !

Example 37.11: A Relativistic Collision



Initially, both protons move in opposite directions, net linear momentum is zero. The three particles after collision are *at rest* again with net linear momentum equals to zero. Thus, relativistic momentum is by design conserved.

We now need to consider the conservation of total relativistic energy.

Question: Find initial velocity of proton.

Example 37.11

Before After

$$2\left(\gamma M_{proton}c^2\right) = 2M_{proton}c^2 + m_{pion}c^2$$

$$2\left(\gamma - 1\right)M_{proton} = m_{pion}$$

$$\gamma = 1 + \frac{m_{pion}}{2M_{proton}} = 1 + \frac{2.40\times10^{-28}\,kg}{2\left(1.67\times10^{-27}\,kg\right)} = 1.072$$

$$\gamma^2 = \frac{c^2}{c^2 - v^2} \quad \rightarrow \quad \gamma^2 c^2 - \gamma^2 v^2 = c^2 \quad \rightarrow \quad v = c\sqrt{1 - 1/\gamma^2}$$

$$v = c\sqrt{1 - 1/1.072^2} = 0.360c$$

Note: Mass *by itself* is not conserved in this process!

New Energy Units

Electron Volt (eV):

The energy equals to moving one positive test charge *e*+ (1 Coulomb) across an electric potential of 1 volt.

$$1eV = (1.6022\times10^{-19}C)(1V) = 1.6022\times10^{-19}J$$

Example: Rest Mass Energy of an electron $m_e = 9.109\times10^{-31}kg$

$$E_0 = m_e c^2 = 9.109\times10^{-31}kg\left(2.997\times10^{8}m/s\right)^2$$

$$= 8.18171\times10^{-14}J\left(\frac{1eV}{1.6022\times10^{-19}J}\right) = 5.11\times10^{5}eV = 0.511MeV$$

$m_e = 0.511MeV/c^2$ ← (mass of *e* in units of *eV* and *c*)

Energy-Momentum Relation

$$E = \gamma mc^2 \qquad p = \gamma mv$$

$$E^2 = \gamma^2 m^2 c^4 \qquad c^2 p^2 = \gamma^2 m^2 v^2 c^2$$

$$E^2 = \frac{m^2 c^4}{1 - v^2/c^2} \qquad c^2 p^2 = \frac{m^2 v^2 c^2}{1 - v^2/c^2}$$

$$\Rightarrow \quad E^2 - c^2 p^2 = m^2 c^4 \left(\frac{1}{1 - v^2/c^2} - \frac{v^2/c^2}{1 - v^2/c^2} \right)$$

$$E^2 - c^2 p^2 = m^2 c^4$$

Energy-Momentum Relation

$$E^2 - c^2 p^2 = m^2 c^4$$

Similar to the space-time interval $ds^2 = dx^2 - c^2 dt^2$ which is *invariant* for *all* inertial observers (independent of relative motion), the combination $E^2 - c^2 p^2$ is also independent of motion and is an *invariant* quantity

→ Both *E* and *P* will change depending on the relative *S*-*S*' velocity
but $E^2 - c^2 p^2$ will not.

Note:
For particles *at rest*, $p = 0$, this expression gives $E = mc^2$ which is the rest mass energy as previously.

For *photons* with no mass, $E=pc$, which can also be shown from Maxwell's Equations.