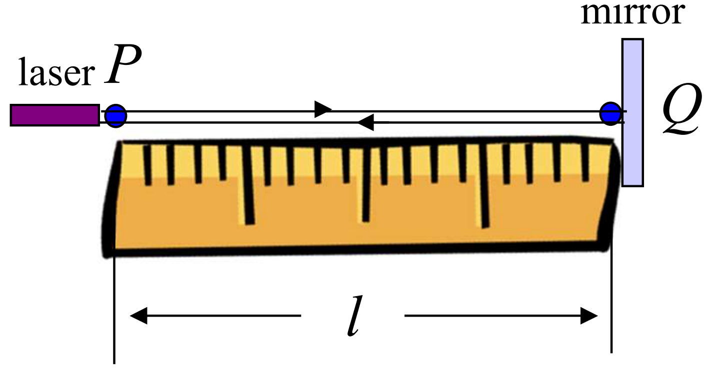
Relativity of Length

→ Distance between two points on a rigid body P & Q can be measured by a light signal's round-trip time.

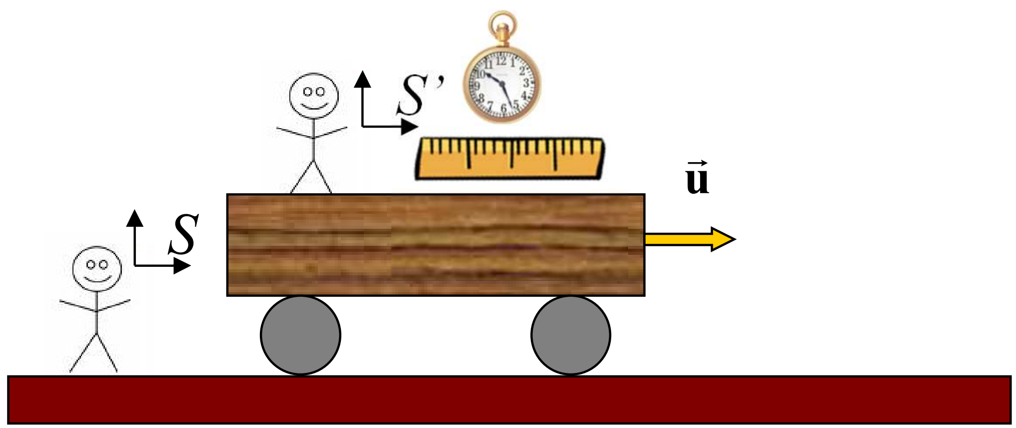


l can be measured by the time interval: $t_2 - t_1$, $2l = c(t_2 - t_1) \quad \rightarrow \quad l = \frac{c}{2}(t_2 - t_1)$

→ As we have seen, Δt will be different for different inertial observers, l will also !

Proper Length

Similar to the concept of **proper time** Δt_0 which is the measured time interval of a clock which is *at rest* with the observer, **proper length** l_0 is the measured length of an object *at rest* with the observer.

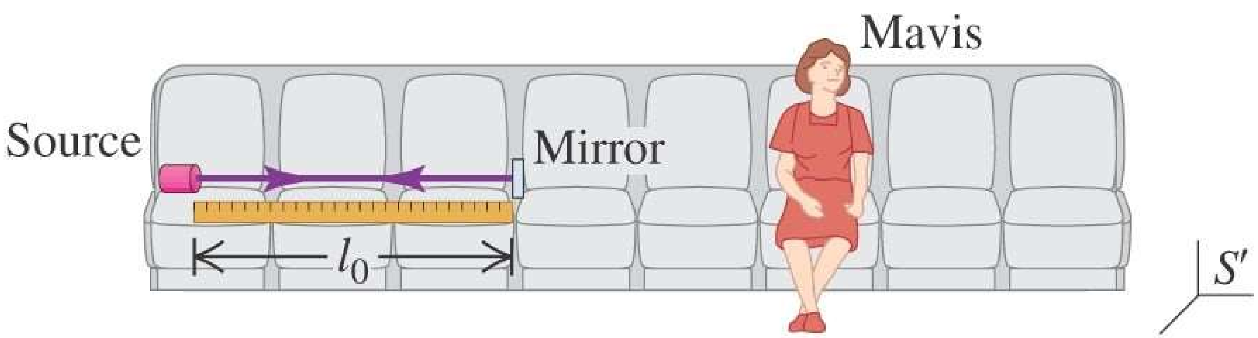


Observer *O*' in *S*'- frame will measure proper time and proper length for the clock and ruler shown.

Length Contraction (parallel to **u**)

Let consider a ruler *at rest* in the moving frame (*S*') and it lays *parallel* to the direction of the relative motion between *S* and *S*' (as shown previously).

Within *S*'- frame, the ruler is at rest with Mavis.



The length of the ruler is measured using a light-clock by measuring the time interval between *two events* (light leaving the laser and light arrives back to the source). In this measurement process, light pulse travels a distance of a total of $2l_0$ within a time interval of Δt_0.

Length Contraction (parallel to **u**)

Since both measurement events are *at rest* (same location) within *S*'-frame, Δt_0 is the *proper time* measurement and we have,

$$\Delta t_0 = \frac{2l_0}{c}$$

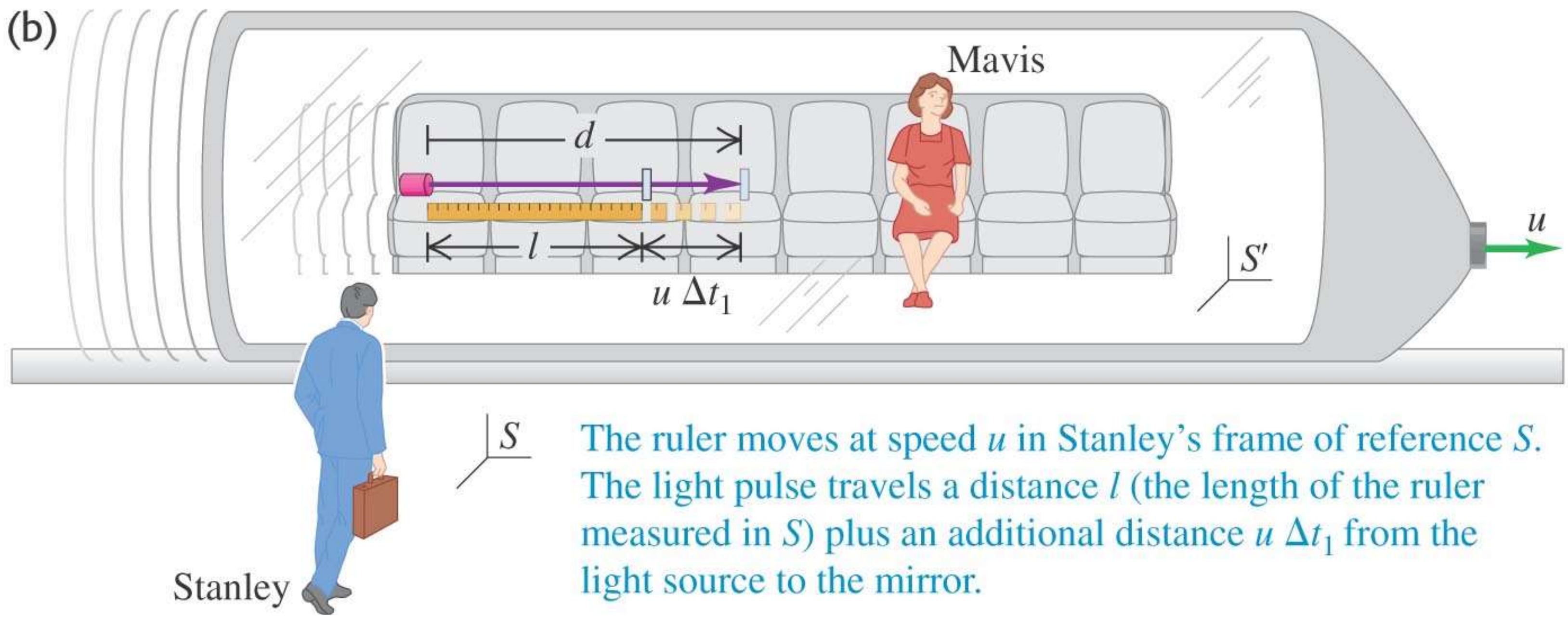
and l_0 is the proper length of the ruler.

Now, let consider the description according to Stanley in *S*-frame.

The ruler according to Stanley will have a length of l and let the time of travel for the light pulse from the source to the mirror be Δt_1.

(note: Since Stanley is measuring these from afar and he is in relative motion with respect to Mavis, his measurement of Δt_1 will NOT be proper.)

Length Contraction (parallel to **u**)



The ruler moves at speed u in Stanley's frame of reference S. The light pulse travels a distance l (the length of the ruler measured in S) plus an additional distance $u\,\Delta t_1$ from the light source to the mirror.

Within the time interval Δt_1, the mirror will have moved a distance of $u\,\Delta t_1$ so that the actual distance that the laser light has to travel is,

$$d = l + u\Delta t_1$$

Since the speed of light is also c in Stanley frame, we can also write,

$$d = c\Delta t_1$$

Length Contraction (parallel to **u**)

Combing these two equations and eliminating d, we have

$$c\Delta t_1 = l + u\Delta t_1$$

$$\Delta t_1 = \frac{l}{c-u}$$

Length Contraction (parallel to **u**)

Combing these two equations by eliminating d, we have

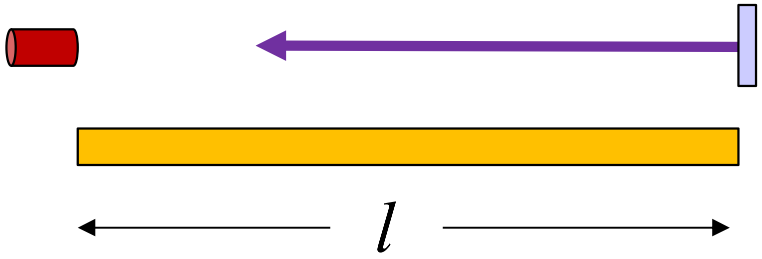
$$c\Delta t_1 = l + u\Delta t_1$$

$$\Delta t_1 = \frac{l}{c-u}$$

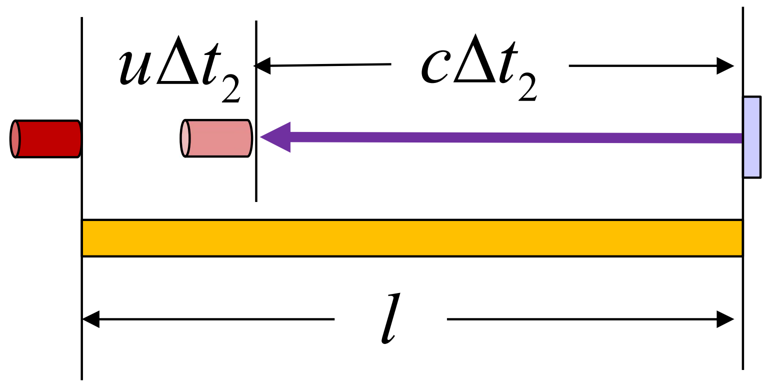
Now, let consider the return trip of the laser light…

Let Δt_2 be the time measured by Stanley for the light to travel back from the mirror back to the source.

Note that as the train moves forward, the source also moves forward to meet the laser light



Length Contraction (parallel to **u**)



Analogous to the outgoing trip, the return trip of the laser light will be *shortened* by $u\Delta t_2$ and

$$c\Delta t_2 = l - u\Delta t_2 \quad \rightarrow \quad \Delta t_2 = \frac{l}{c+u}$$

The total travel time for the laser pulse is $\Delta t = \Delta t_1 + \Delta t_2$

$$\Delta t = \frac{l}{c-u} + \frac{l}{c+u} = \frac{l(c \cancel{+u} + c \cancel{-u})}{c^2 - u^2} = \frac{2lc}{c^2\left(1-u^2/c^2\right)} = \frac{2l}{c\left(1-u^2/c^2\right)}$$

Length Contraction (parallel to **u**)

From the time dilation formula, we also have

$$\Delta t = \gamma\, \Delta t_0$$

Combing these two equations for Δt, we have

$$\frac{2l}{c\left(1-u^2/c^2\right)} = \frac{1}{\left(1-u^2/c^2\right)^{1/2}}\frac{2l_0}{c}$$

$$\frac{l}{\left(1-u^2/c^2\right)^{1/2}} = l_0$$

or $$\boxed{l = \frac{l_0}{\gamma}}$$ (length contraction)

Recall: The two timing events for the length measurement are *at rest* in *S*'-frame so that Mavis measure the *proper time* Δt_0 and *proper length* l_0.

On the other hand, Stanley's measurements (Δt and l) of the same two events are *not proper*.

Note: The proper length l_0 is always the *longest* among all inertial observers.

Unreality Check

According to Galilean Velocity Transformation, in *S*-frame, we have

On the out going trip, $c_1 = c + u$ so that $\Delta t_1 = \frac{l}{c_1 - u} = \frac{l}{c + u - u} = \frac{l}{c}$

On the returning trip, $c_2 = c - u$ so that $\Delta t_2 = \frac{l}{c_2 + u} = \frac{l}{c - u + u} = \frac{l}{c}$

Now, the total time for the whole trip is now,

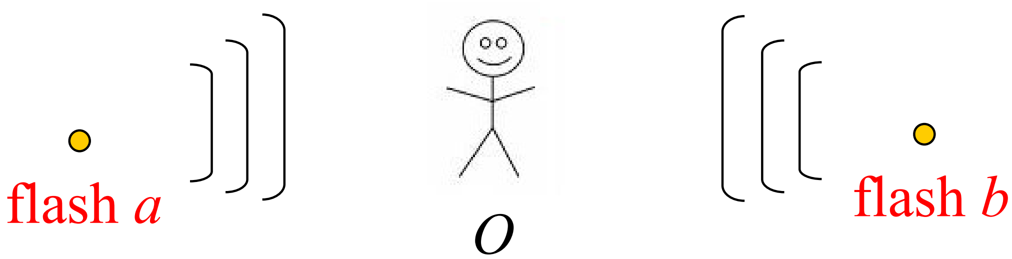
$$\Delta t = \Delta t_1 + \Delta t_2 = \frac{2l}{c}$$

From our previous "unreality check", $\Delta t = \Delta t_0$ (all clocks clicks at the same rate)

$$\Rightarrow \quad \Delta t = \frac{2l}{c} \Leftrightarrow \frac{2l_0}{c} = \Delta t_0 \quad \rightarrow \quad l = l_o$$

Relativity of Simultaneity

Simultaneity

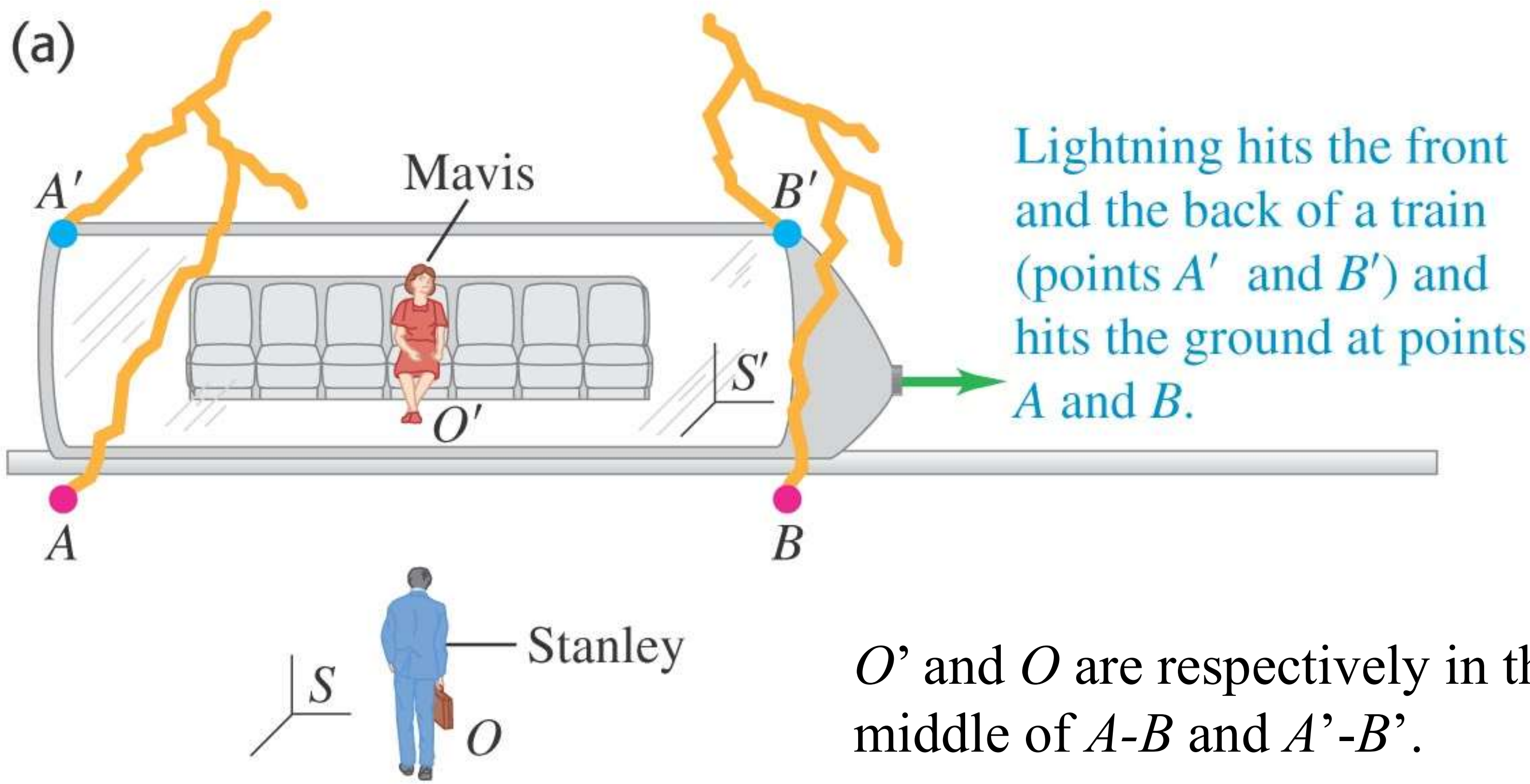


Definition: Two flashes (event *a* and *b*) are considered to be simultaneous with respect to observer *O* if light from *a* & *b* (equal distance to observer *O* in the middle) arrive at the observer at the same time.

Relativity of Simultaneity

We will analyze the situation in Stanley's *S*-frame in the following slides.

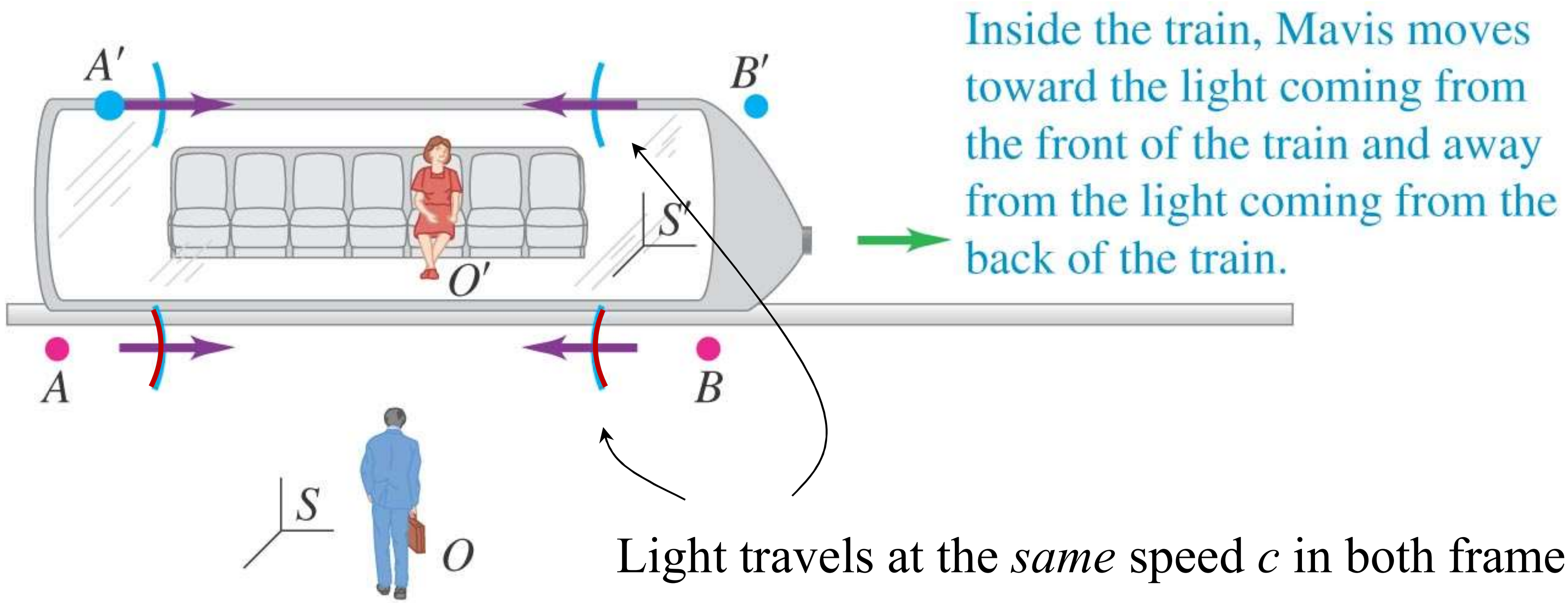
At time *t* = 0:



O' and *O* are respectively in the middle of *A*-*B* and *A*'-*B*'.

Relativity of Simultaneity

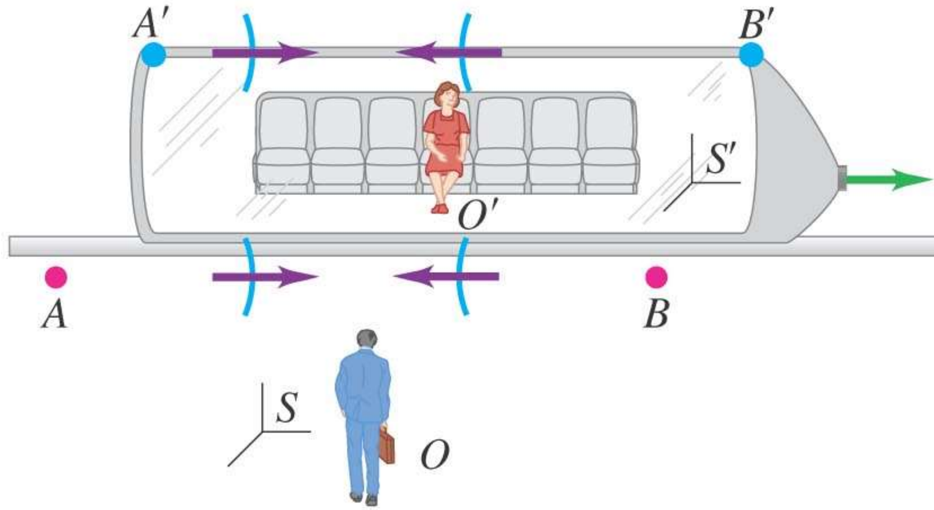
On the ground (*S*-frame) after some time *t*,



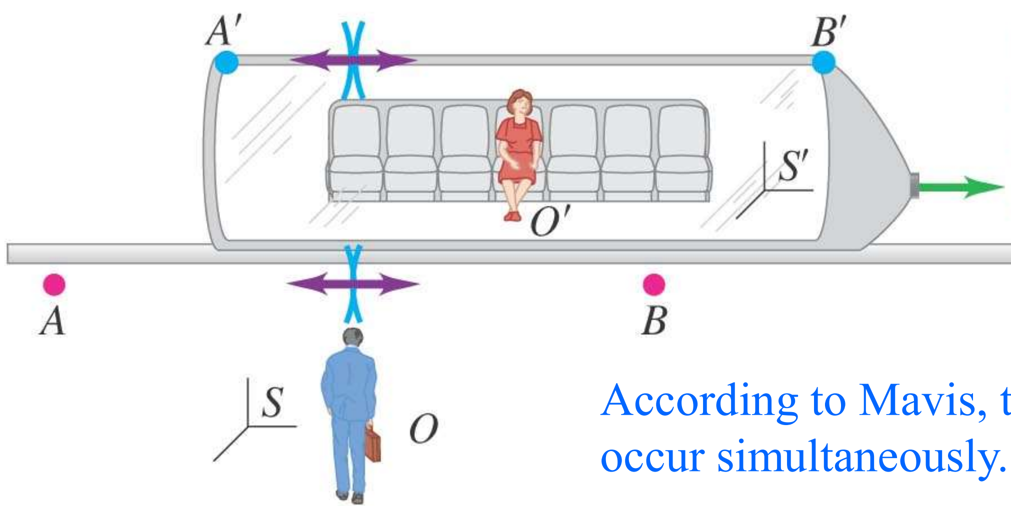
Light travels at the *same* speed *c* in both frames.

Relativity of Simultaneity

On the ground (*S*-frame) after some more time *t*,



Because Mavis sees the light at the front of the train first, she believes that the lightning at the front of the train hit first.



Then, a bit later, when Stanley see the lights from *A* and *B* reach him *simultaneously*, Mavis in *S*'- frame still waiting for the light from the lighting strike at the rear of the train to reach her.

According to Mavis, the lighting strikes at *A*' and *B*' did not occur simultaneously. Stanley agrees with Mavis' observation.

Unreality Check

In the (incorrect) Galilean view, *according to Stanley*, $c_{A'}$ and $c_{B'}$ inside the train will be modified according to their relative speed u.

So that light signals from *A*' will speed up and signal from *B*' will slow down in Stanley's viewpoint.

$$\left.\begin{aligned} c_{A'} &= c+u \\ c_{B'} &= c-u \end{aligned}\right\}$$

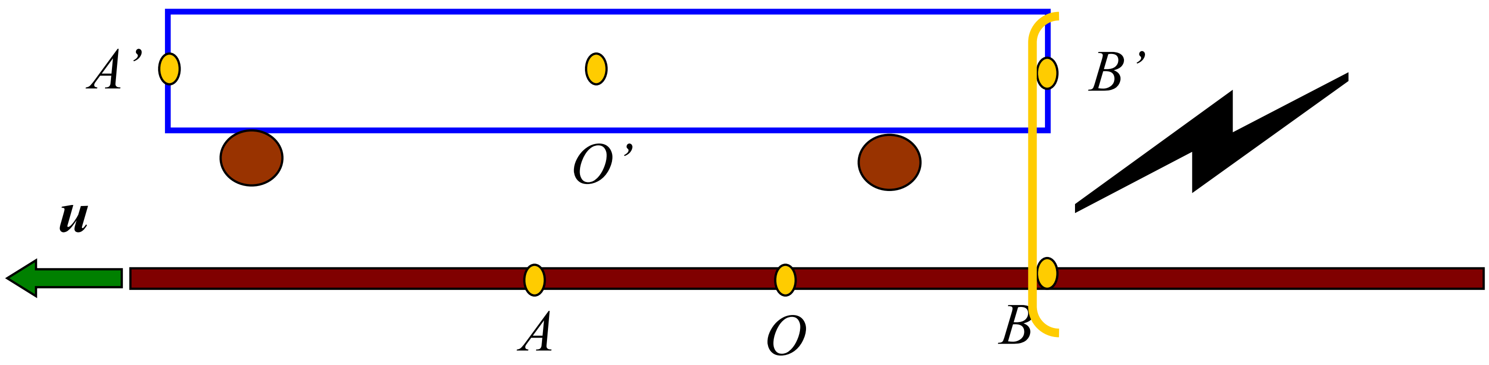
Speed of light from *A*' and *B*' will be modified by the box car's speed u.

This changes in speed for c will result in signals from *A*' and *B*' to arrive at *O*' at the same time so that Stanley will say that *both* Marvis and himself will observe the lighting strikes simultaneously.

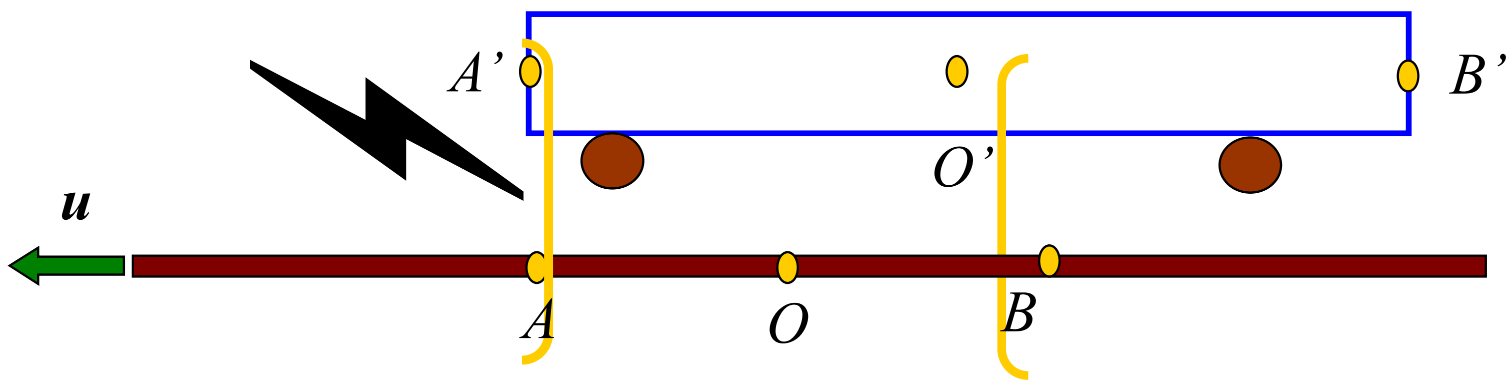
Sequence of Events according to Mavis (*S'*- frame) [box car *stationary*]

At *t'* = 0, lightning strikes on *B* and *B'*:

Note:
1. platform is length contracted and the box car's rest length is longer.
2. Speed of light is the same in both frame according to *O'*.

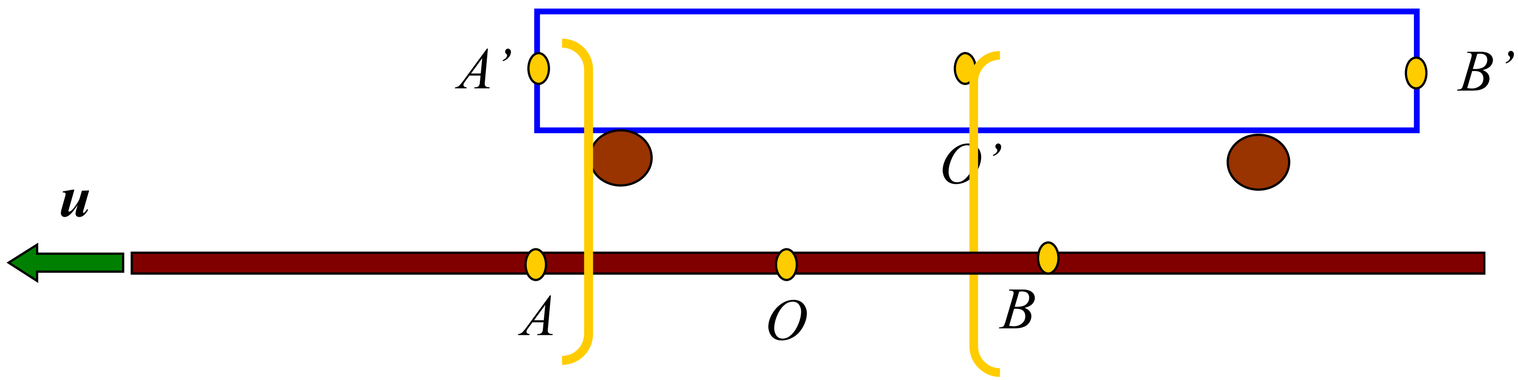


Some time later, lightning strikes on *A* and *A'*:



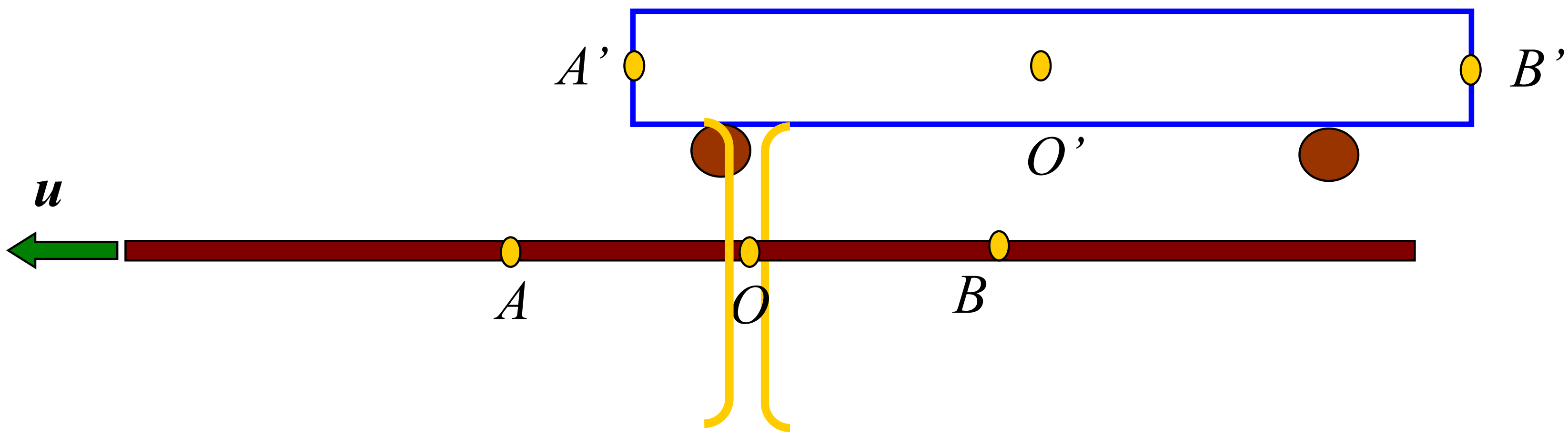
Sequence of Events according to Mavis (*S*'- frame)

Some time later, the light pulses reach *O'*



Sequence of Events according to Mavis (*S'*- frame)

Some more time later, the light pulses reach *O* at the same time but light pulse from A' still has not reach O' yet.

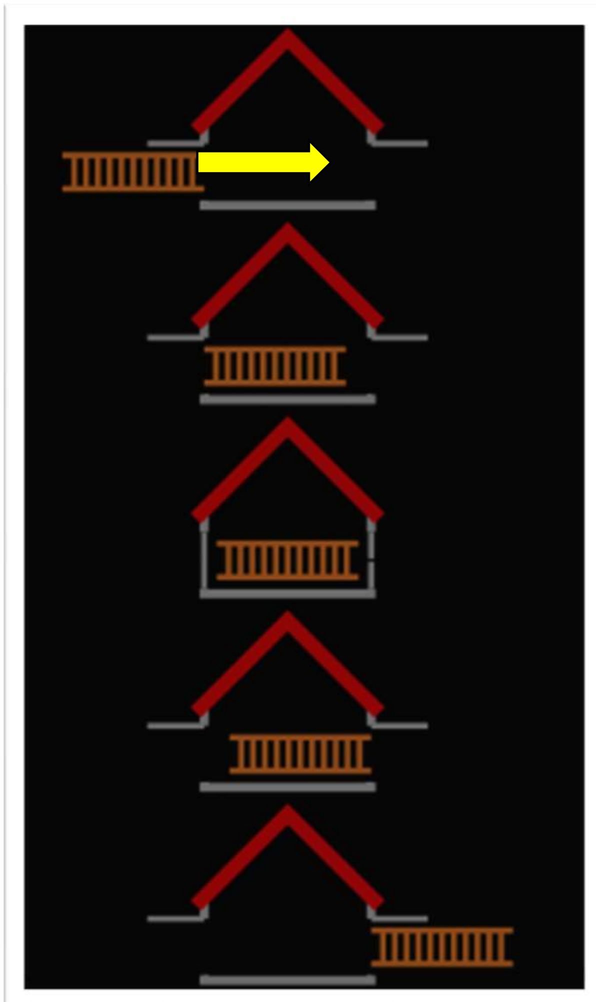


Mavis see the flashes at *different* time but agrees with Stanley that the light signals in Stanley's frame arrive at him at the same time. Both see the same sequence of events but the notion of simultaneity is relative.

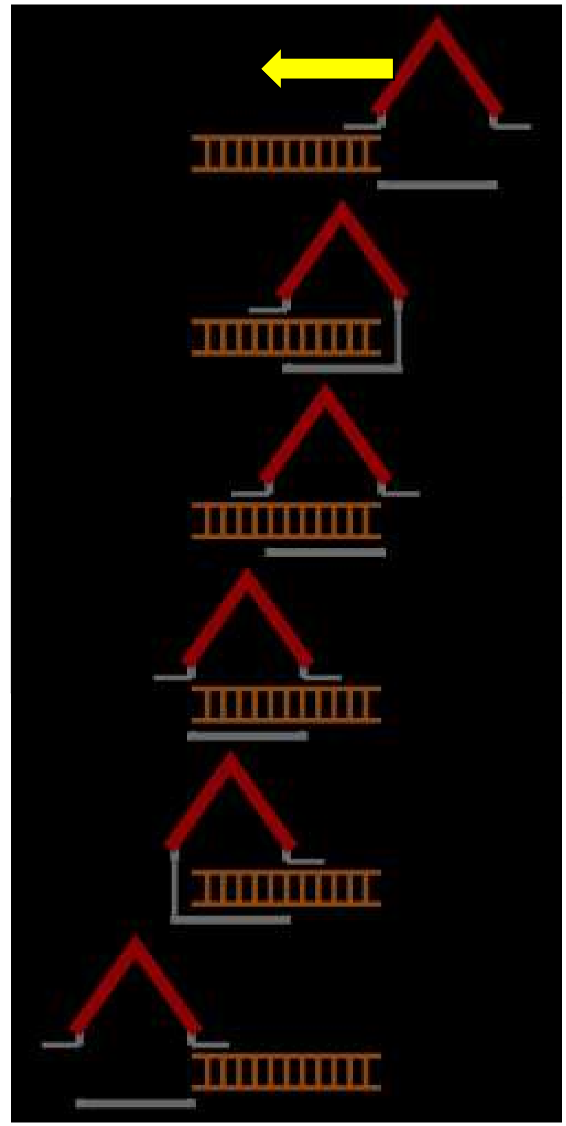
Ladder | Barn Paradox

Pictures from wikipedia

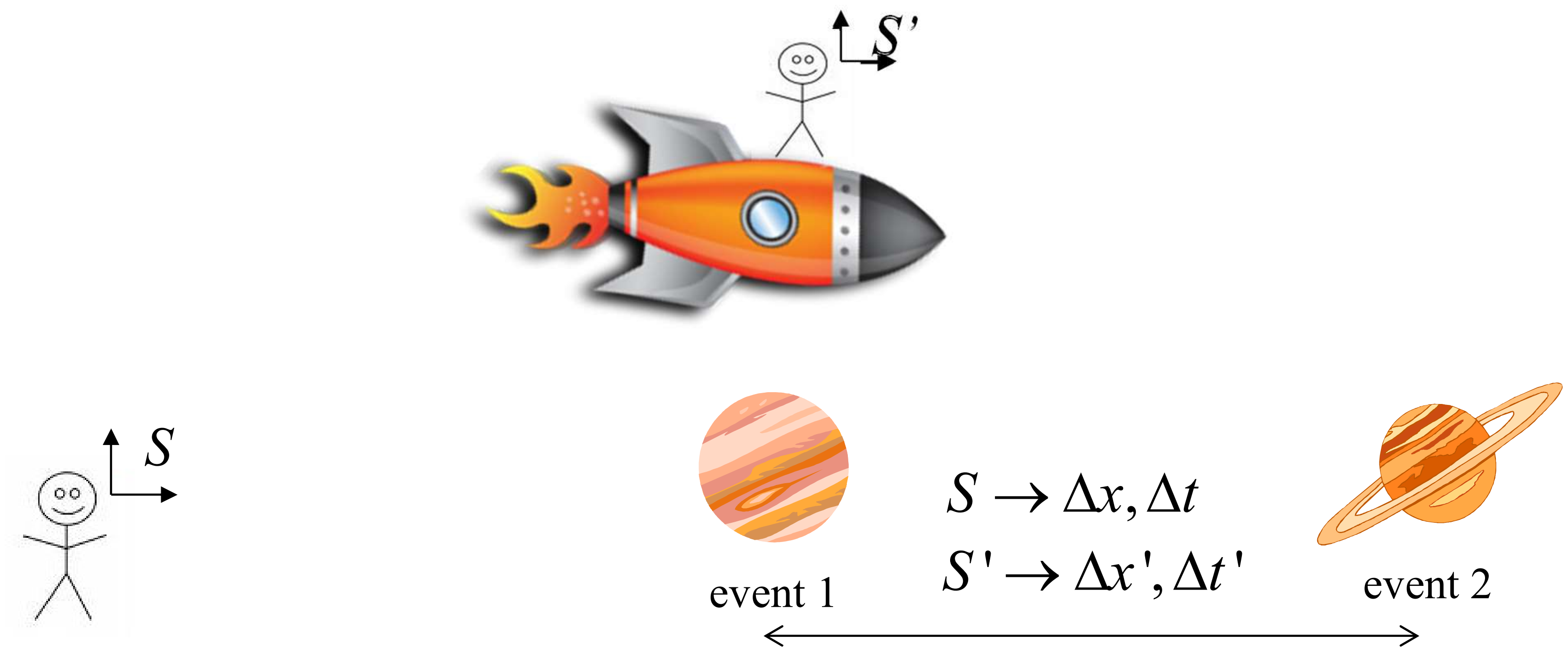
Barn's Frame



Ladder's Frame



Relative Speed of a Spaceship



$\Delta x \rightarrow proper$

$\Delta x' \rightarrow \text{contracted}$

$\Delta x' = \Delta x / \gamma$

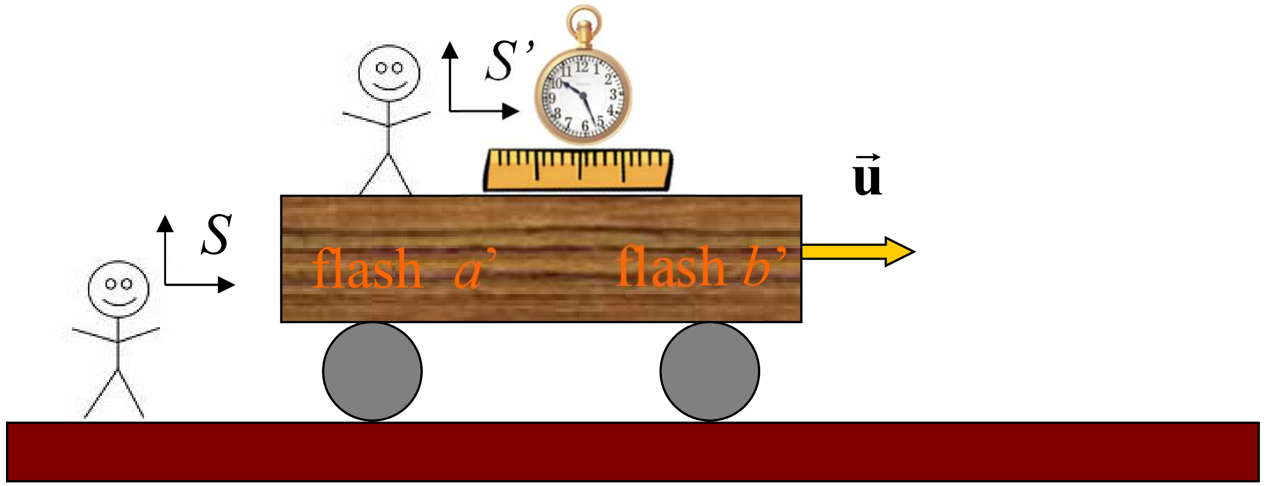
$\Delta t \rightarrow dilated$

$\Delta t' \rightarrow proper$

$\Delta t = \gamma \Delta t'$

$$u' = \frac{\Delta x'}{\Delta t'} = \frac{\Delta x / \gamma}{\Delta t / \gamma} = \frac{\Delta x}{\Delta t} = u$$

Summary of Results



Time Dilation: (moving clock runs slow)

$$\Delta t = \gamma\, \Delta t_0, \quad \gamma = 1\Big/\sqrt{1-u^2/c^2}$$

Measured by S

Measured by S'

$$L = L_0/\gamma$$

Length Contraction: (moving ruler get shorter)

Δt_0 is measured at the same location wrt S'

l_0 is measured at the same time wrt S'

Numerical Notes

1. Keep u in units of c, e.g., $u = 0.8c$ since we always have the ratio u/c. Then,

$$\gamma = 1\Big/\sqrt{1-(0.8c)^2/c^2} = 1\Big/\sqrt{1-(0.8)^2}$$ can be calculated simply.

2. Since the ratio u/c occurs so often, it has a special symbol:

$$\beta = u/c \quad \text{and} \quad \gamma = 1\Big/\sqrt{1-\beta^2}$$

3. Approximation when $u \ll c$ (*low* speed: *non-relativistic*):

Recall Binominal Expansion:

$$(1+\varepsilon)^n \cong 1 + n\varepsilon \quad \text{for } \varepsilon \textit{ small}.$$

Numerical Notes

3. Approximation when $u << c$ (*low* speed: *non-relativistic*):

$$\gamma = 1\Big/\sqrt{1-u^2/c^2}$$
$$= \left(1-u^2/c^2\right)^{-1/2}$$
$$\cong 1+\left(-\frac{1}{2}\right)\left(-\frac{u^2}{c^2}\right) = 1+\frac{u^2}{2c^2}$$

$(1+\varepsilon)^n \cong 1+n\varepsilon$

If u/c is small, $u^2/2c^2$ will be really small !

Problem: calculator might treat $1+u^2/2c^2$ simply as 1 and you will loose all significant figures if you *combine* 1 and $u^2/2c^2$.

Relativistic (u=0.3c) :

$u^2/2c^2 = 0.3^2/2 = 0.045$

$\gamma = 1.24$ (quite diff. from 1)

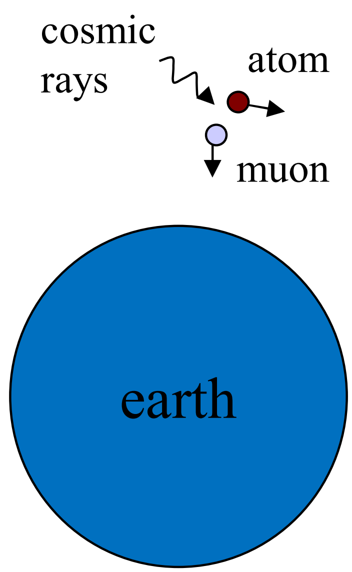
Non-Relativistic (u=0.003c) :

$u^2/2c^2 = 0.03^2/2 = 0.0000045$

$\gamma = 1.0000045$ (not much diff. from 1)

Typically, if $u \geq 0.1c$, one should consider speed to be *relativistic*.

Example: Muon Life Time



Muon is an unstable elementary particle with:

- e-charge
- 207 M_e

Produced by cosmic rays (mostly high energy protons) collide with atoms high up in the atmosphere (>4 km).

In a the *rest frame* of the muon, it has an average life-time of:

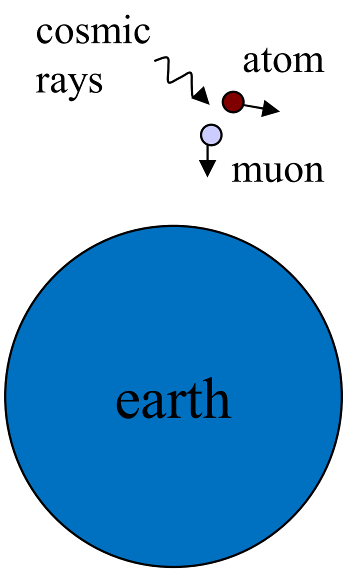
$$\tau_{muon} = 2.2\mu s$$

Without Einstein's SR, if $u_{muon} \cong 0.99c$

$$\text{avg. dist. travelled} = 0.99(3\times10^{8}m/s)(2.2\times10^{-6}s)$$

$$\cong 600m$$

Example: Muon Life Time



Muon is typically produced high up in the atmosphere ~ 4 to 13 km.

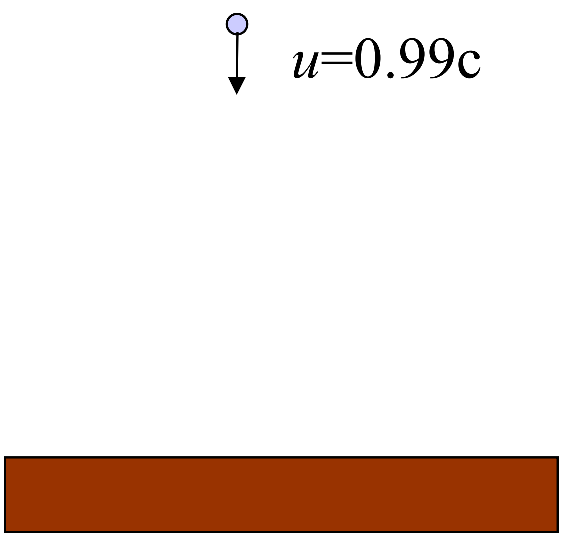
So, according to Galilean relativity, muon can't reach the surface of the earth & we should not be able to detect them !

However, we do detect cosmic rays muons on the ground !

Einstein's Relativity resolves this apparent paradox…

Example: Muon Life Time

Earth's frame:



Earth is stationary & muon move downward toward the ground

The two events (creation and decay) that define the *life-time* of a muon occurs in the *rest frame* of the muon so that the 2.2μs life-time is the **proper time** for an observer in muon's rest frame: $\Delta t_0 = 2.2\mu s$

In contrast, in *Earth's frame*, these two events do not occur at rest w/ Earth and the interval (Δt) defined by these two events will *not* be a *proper* time interval as observed by someone in the Earth's frame. In fact, Δt will be dilated:

$$\Delta t = \gamma\,\Delta t_0 = \frac{\Delta t_0}{\sqrt{1-u^2/c^2}} = \frac{2.2\mu s}{\sqrt{1-0.99^2}} = 16\mu s$$

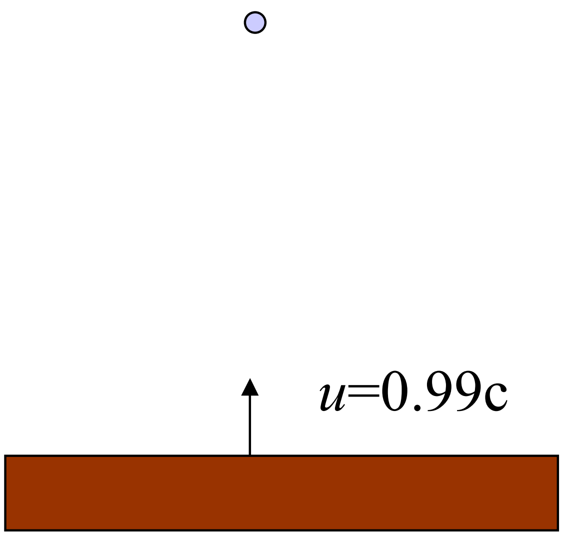
Example: Muon Life Time

Earth's frame: So, according to an Earth's observer, avg. dist. travelled by these cosmic rays muons will be,

$$\text{avg. dist. travelled} = (16\mu s)(0.99)(3\times 10^{8} m/s) = 4800m$$

So, some muons will reach the surface of the earth !

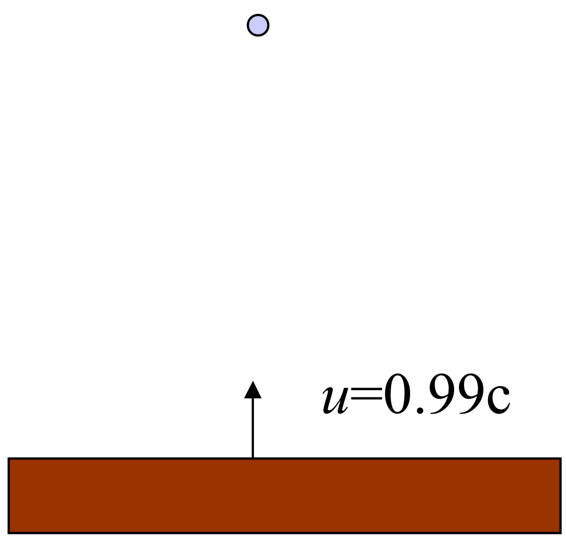
Muon's frame: Muon is stationary & the Earth moves upward at 0.99c.



In the Earth's rest frame, the atmosphere is at rest with the Earth and its height (l_0=4000m) is the **proper length** measured by an earth observer.

Example: Muon Life Time

Muon's frame:



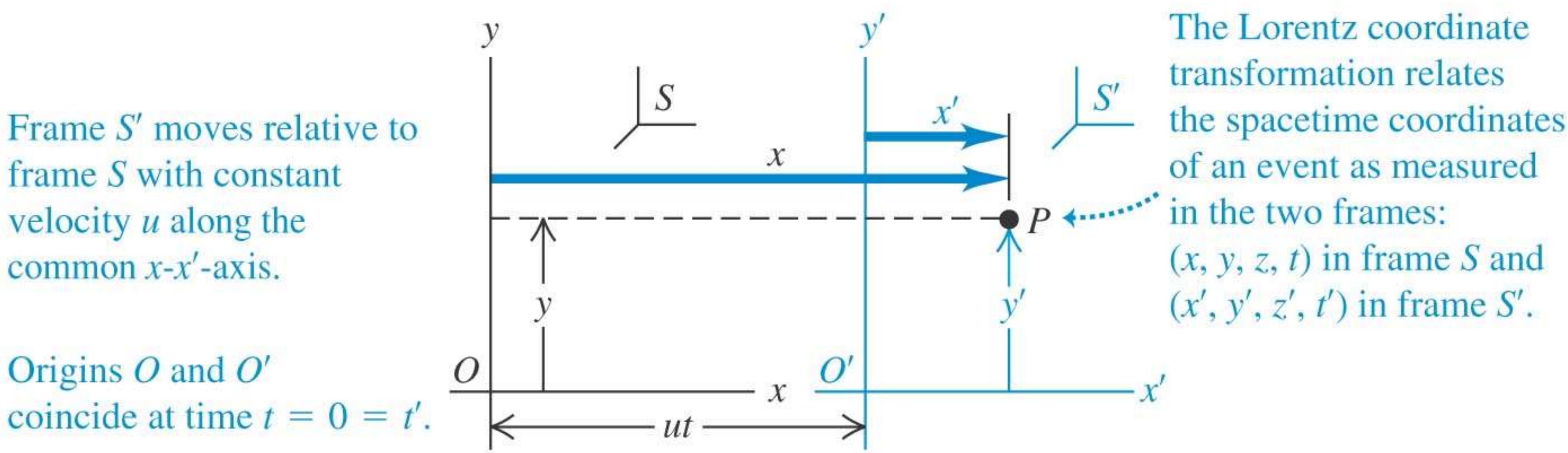
But in muon's viewpoint, the Earth is moving toward it at 0.99c and the height of the atmosphere will be length contracted in muon's frame !

$$l = \frac{l_0}{\gamma} = \sqrt{1-0.99^2}\,(4000m) \cong 560m < 600m$$

So, on average, some muons will be able to reach the surface of the Earth before it decays.

Lorentz Coordinate Transformation

Transforming the space-time coordinates from S to S' correctly so that physical laws satisfying SR are invariant.



u in x-dir only.

$$(x, y, z, t) \quad \longleftrightarrow \quad (x', y', z', t')$$

Lorentz Transformation

$$S \to S' \begin{cases} x' = \gamma(x - ut) \\ y' = y \\ z' = z \\ t' = \gamma\left(t - \dfrac{u}{c^2}x\right) \end{cases} \qquad S' \to S \begin{cases} x = \gamma(x' + ut') \\ y = y' \\ z = z' \\ t = \gamma\left(t' + \dfrac{u}{c^2}x'\right) \end{cases}$$

Since *O* and *O*' are in constant relative motion, the physical description and coordinate transformation between them should be symmetric !

$\Rightarrow$ with $u \to -u$

Lorentz Transformation (derivation)

Assume a modification of the Galilean Transformation by a still to be determined correction factor γ:

$$x = \gamma(x' + ut')$$

Since *S* and *S*' are in relative motion, we should have a symmetric equation for *x*' with $u \rightarrow -u$,

$$x' = \gamma(x - ut)$$

Now, set both *S* and *S*' to coincide with each other at the origin at $t = t' = 0$ and a light pulse is initiated at that time.

Lorentz Transformation (derivation)

After a time t in S and a corresponding t' in S',

$$\left.\begin{aligned} x &= ct \quad \text{in S} \\ x' &= ct' \quad \text{in S'} \end{aligned}\right\}$$

c is the same in both frames!

Substitute these into the previous equations, we have:

$$\left.\begin{aligned} x &= \gamma(x' + ut') \\ x' &= \gamma(x - ut) \end{aligned}\right\} \rightarrow \begin{aligned} ct &= \gamma(ct' + ut') = \gamma(c+u)t' \\ ct' &= \gamma(ct - ut) \end{aligned}$$

Substitute t' from the bottom equation to the top equation, we have:

$$c\cancel{t} = \gamma(c+u)\left(\frac{\gamma}{c}(c-u)\cancel{t}\right) \qquad \gamma^2 = \frac{c^2}{c^2 - u^2} = \frac{1}{1 - u^2/c^2}$$

$$c^2 = \gamma^2(c^2 - u^2) \qquad \gamma = \frac{1}{\sqrt{1 - u^2/c^2}}$$

(the desired γ factor)

Lorentz Transformation (derivation)

Now we try to eliminate the x variable in the original two equations:

$$x' = \gamma(x - ut) \qquad x = \gamma(x' + ut')$$

$$x' = \gamma(\gamma(x' + ut') - ut)$$

$$x' = \gamma^2 x' + \gamma^2 ut' - \gamma ut$$

$$\gamma ut = \gamma^2 ut' + (\gamma^2 - 1)x'$$

$$t = \gamma t' + \frac{\gamma^2 - 1}{\gamma u} x' = \gamma\left(t' + \frac{\gamma^2 - 1}{\gamma^2 u} x'\right)$$

Expanding out the γ factor on the RHS of the expression,

$$\gamma = \frac{1}{\sqrt{1 - u^2/c^2}} = \frac{c}{\sqrt{c^2 - u^2}} \quad \rightarrow \quad \gamma^2 = \frac{c^2}{c^2 - u^2}$$

Lorentz Transformation (derivation)

More algebra…,

$$\gamma^2 - 1 = \frac{c^2}{c^2 - u^2} - 1 = \frac{c^2 - c^2 + u^2}{c^2 - u^2} = \frac{u^2}{c^2 - u^2}$$

$$\frac{\gamma^2 - 1}{\gamma^2 u} = \frac{u^{\cancel{2}}}{\cancel{c^2 - u^2}}\left(\frac{\cancel{c^2 - u^2}}{c^2}\right)\frac{1}{\cancel{u}} = \frac{u}{c^2}$$

$$\Rightarrow \quad t = \gamma\left(t' + \frac{\gamma^2 - 1}{\gamma^2 u}x'\right) = \gamma\left(t' + \frac{u}{c^2}x'\right)$$

(the desired Lorentz Transform in time)

Lorentz Transformation

$$S \rightarrow S' \begin{cases} x' = \gamma(x - ut) \\ y' = y \\ z' = z \\ t' = \gamma\left(t - \dfrac{u}{c^2}x\right) \end{cases}$$

$$S' \rightarrow S \begin{cases} x = \gamma(x' + ut') \\ y = y' \\ z = z' \\ t = \gamma\left(t' + \dfrac{u}{c^2}x'\right) \end{cases}$$

Lorentz Transformation

Reduction back to Galilean Transformation in the regime: $u \ll c$.

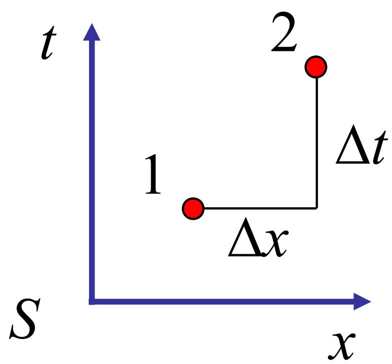
$$\text{For } \frac{u}{c} \ll 1 \quad \rightarrow \quad \gamma = \frac{1}{\sqrt{1 - u^2/c^2}} \cong 1$$

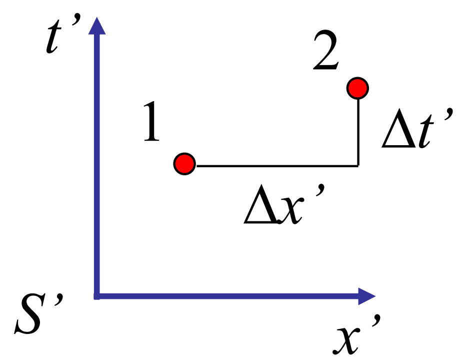
$$\Rightarrow \begin{cases} x' \cong 1\left(x - ut\right) = x - ut \\ t' = 1\left(t - \cancel{\dfrac{u}{c^2}} x\right) = t \end{cases} \qquad \text{(Galilean Transformation)}$$

Einstein Relativity is more general and it reduces to previous results (Galilean) in the $u \ll c$ limit.

Intervals between Two Events

We have defined Lorentz Transformation for an event *P*. Now, we want to extend it to a *space-time interval* (Δx, Δt) between two events:





Events 1 and 2 will have different coordinates for different observers *O* and *O*'.

$$\Delta x = x_2 - x_1,\ \Delta t = t_2 - t_1$$

$$\Delta x' = x'_2 - x'_1,\ \Delta t' = t'_2 - t'_1$$

In general, $\Delta x \neq \Delta x' \quad and$

$$\Delta t \neq \Delta t'$$

Intervals between Two Events

By direct substitution, we have the following transform for intervals between two events:

$$\begin{cases} \Delta x = \gamma\left(\Delta x' + u\Delta t'\right) \\ \Delta t = \gamma\left(\Delta t' + \dfrac{u}{c^2}\Delta x'\right) \end{cases}$$

or,

$$\begin{cases} dx = \gamma\left(dx' + u\,dt'\right) \\ dt = \gamma\left(dt' + \dfrac{u}{c^2}dx'\right) \end{cases}$$

for differential changes

Note: One can show that the combination $ds^2 = dx^2 - c^2 dt^2 = ds'^2 = dx'^2 - c^2 dt'^2$ is the same (*invariant*) for all inertial observation.