Numerical Notes

1. Keep u in units of c, e.g., $u = 0.8c$ since we always have the ratio u/c. Then,

$$\gamma = 1\Big/\sqrt{1-(0.8c)^2/c^2} = 1\Big/\sqrt{1-(0.8)^2}$$ can be calculated simply.

2. Since the ratio u/c occurs so often, it has a special symbol:

$$\beta = u/c \quad \text{and} \quad \gamma = 1\Big/\sqrt{1-\beta^2}$$

3. Approximation when $u \ll c$ (*low* speed: *non-relativistic*):

Recall Binominal Expansion:

$$(1+\varepsilon)^n \cong 1 + n\varepsilon \qquad \text{for } \varepsilon \text{ small.}$$

Numerical Notes

3. Approximation when $u << c$ (*low* speed: *non-relativistic*):

$$\gamma = 1\Big/\sqrt{1-u^2/c^2}$$
$$= \left(1-u^2/c^2\right)^{-1/2}$$
$$\cong 1+\left(-\frac{1}{2}\right)\left(-\frac{u^2}{c^2}\right) = 1+\frac{u^2}{2c^2}$$

$(1+\varepsilon)^n \cong 1+n\varepsilon$

If u/c is small, $u^2/2c^2$ will be really small !

Problem: calculator might treat $1+u^2/2c^2$ simply as 1 and you will loose all significant figures if you *combine* 1 and $u^2/2c^2$.

Relativistic (u=0.3c) :

$$u^2/2c^2 = 0.3^2/2 = 0.045$$

$\gamma = 1.24$ (quite diff. from 1)

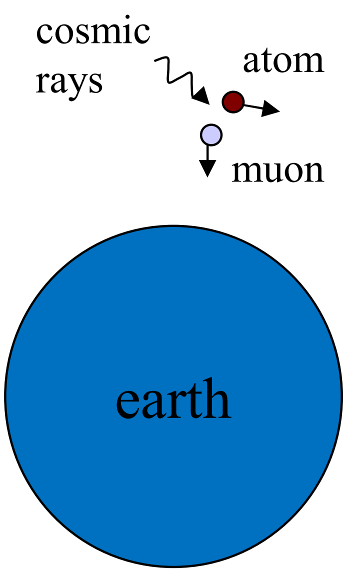
Non-Relativistic (u=0.003c) :

$$u^2/2c^2 = 0.03^2/2 = 0.0000045$$

$\gamma = 1.0000045$ (not much diff. from 1)

Typically, if $u \geq 0.1c$, one should consider speed to be *relativistic*.

Example: Muon Life Time



Muon is an unstable elementary particle with:

- e-charge
- 207 M_e

Produced by cosmic rays (mostly high energy protons) collide with atoms high up in the atmosphere (>4 km).

In a the *rest frame* of the muon, it has an average life-time of:

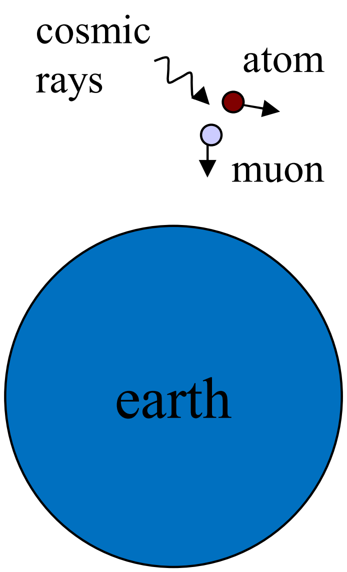
$$\tau_{muon} = 2.2\mu s$$

Without Einstein's SR, if $u_{muon} \cong 0.99c$

$$\text{avg. dist. travelled} = 0.99(3\times10^{8}\,m/s)(2.2\times10^{-6}\,s)$$

$$\cong 600m$$

Example: Muon Life Time



Muon is typically produced high up in the atmosphere ~ 4 to 13 km.

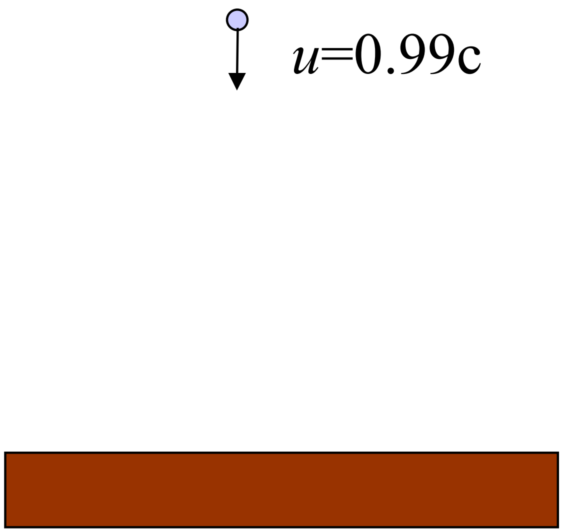
So, according to Galilean relativity, muon can't reach the surface of the earth & we should not be able to detect them !

However, we do detect cosmic rays muons on the ground !

Einstein's Relativity resolves this apparent paradox…

Example: Muon Life Time

Earth's frame:



Earth is stationary & muon move downward toward the ground

The two events (creation and decay) that define the *life-time* of a muon occurs in the *rest frame* of the muon so that the 2.2μs life-time is the **proper time** for an observer in muon's rest frame: $\Delta t_0 = 2.2\mu s$

In contrast, in *Earth's frame*, these two events do not occur at rest w/ Earth and the interval (Δt) defined by these two events will *not* be a *proper* time interval as observed by someone in the Earth's frame. In fact, Δt will be dilated:

$$\Delta t = \gamma\,\Delta t_0 = \frac{\Delta t_0}{\sqrt{1-u^2/c^2}} = \frac{2.2\mu s}{\sqrt{1-0.99^2}} = 16\mu s$$

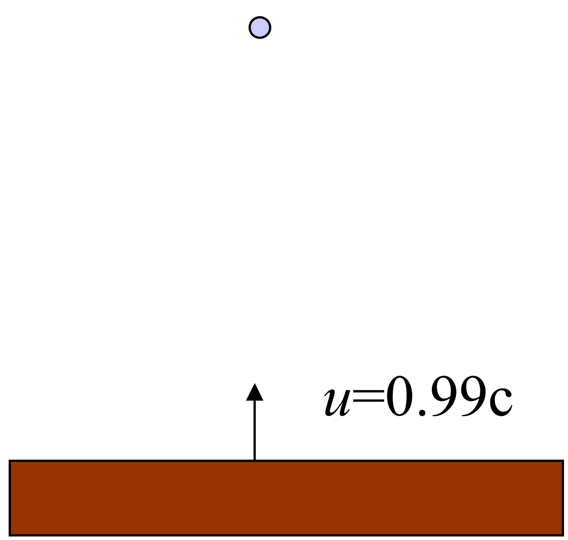
Example: Muon Life Time

Earth's frame: So, according to an Earth's observer, avg. dist. travelled by these cosmic rays muons will be,

$$\text{avg. dist. travelled} = (16\mu s)(0.99)(3\times10^{8}\, m/s) = 4800m$$

So, some muons can reach the surface of the earth !

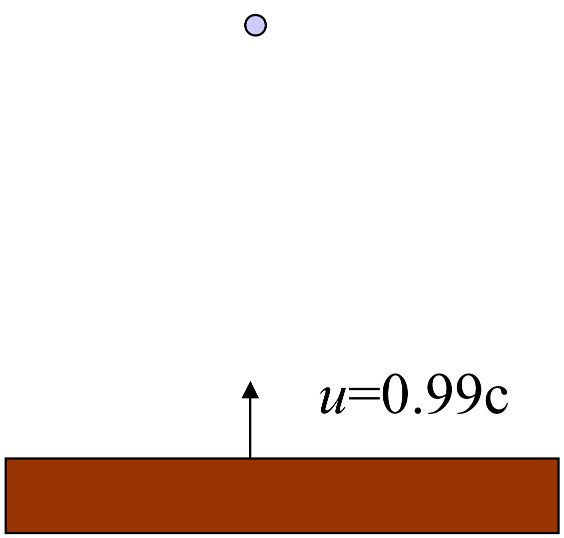
Muon's frame: Muon is stationary & the Earth moves upward at 0.99c.



In the Earth's rest frame, the atmosphere is at rest with the Earth and its height (l_0=4000m) is the **proper length** measured by an earth observer.

Example: Muon Life Time

Muon's frame:



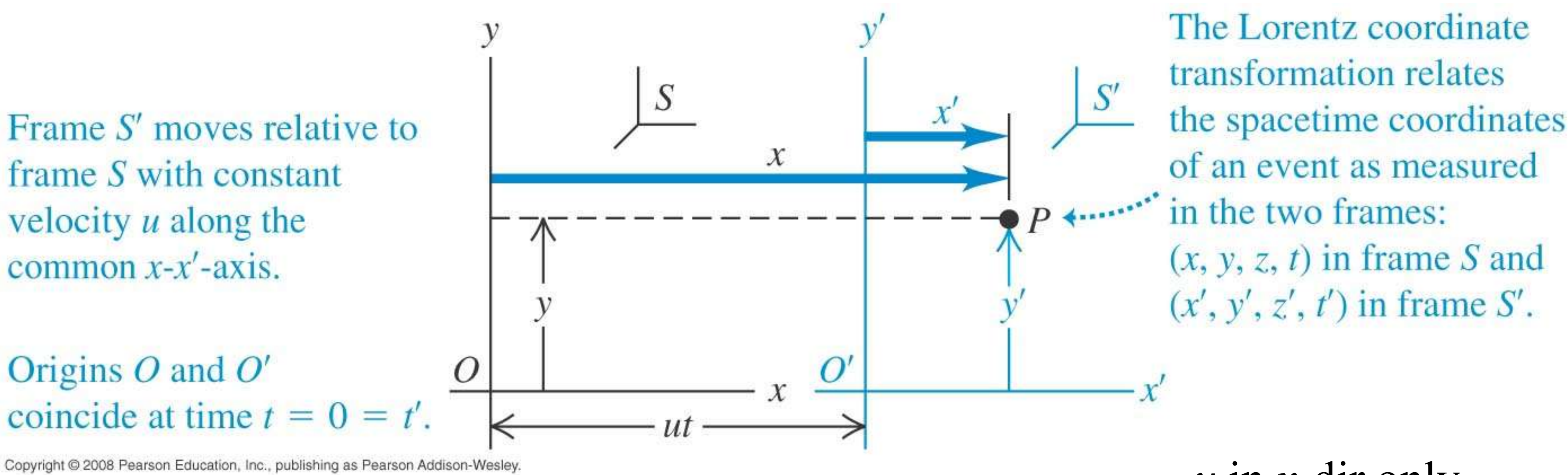
But in muon's viewpoint, the Earth is moving toward it at 0.99c and the height of the atmosphere will be length contracted in muon's frame !

$$l = \frac{l_0}{\gamma} = \sqrt{1 - 0.99^2}\,(4000m) \cong 560m < 600m$$

So, on average, some muons will be able to reach the surface of the Earth before it decays.

Lorentz Coordinate Transformation

Transforming the space-time coordinates from *S* to *S'* correctly so that physical laws satisfying SR are invariant.



u in *x*-dir only.

$$(x, y, z, t) \quad \longleftrightarrow \quad (x', y', z', t')$$

Lorentz Transformation

$$S \rightarrow S' \begin{cases} x' = \gamma(x - ut) \\ y' = y \\ z' = z \\ t' = \gamma\left(t - \dfrac{u}{c^2} x\right) \end{cases} \qquad S' \rightarrow S \begin{cases} x = \gamma(x' + ut') \\ y = y' \\ z = z' \\ t = \gamma\left(t' + \dfrac{u}{c^2} x'\right) \end{cases}$$

Since *O* and *O*' are in constant relative motion, the physical description and coordinate transformation between them should be symmetric !

⇨ with $u \rightarrow -u$

Lorentz Transformation (derivation)

Assume a modification of the Galilean Transformation by a still to be determined correction factor γ:

$$x = \gamma(x' + ut')$$

Since S and S' are in relative motion, we should have a symmetric equation for x' with $u \rightarrow -u$,

$$x' = \gamma(x - ut)$$

Now, set both S and S' to coincide with each other at the origin at $t = t' = 0$ and a light pulse is initiated at that time.

Lorentz Transformation (derivation)

After a time t in S and a corresponding t' in S',

$$\left.\begin{aligned} x &= ct \quad \text{in S} \\ x' &= ct' \quad \text{in S'} \end{aligned}\right\}$$

c is the same in both frames!

Substitute these into the previous equations, we have:

$$\left.\begin{aligned} x &= \gamma(x'+ut') \\ x' &= \gamma(x-ut) \end{aligned}\right\} \rightarrow \begin{aligned} ct &= \gamma(ct'+ut') = \gamma(c+u)t' \\ ct' &= \gamma(ct-ut) \end{aligned}$$

Substitute t' from the bottom equation to the top equation, we have:

$$c\cancel{t} = \gamma(c+u)\left(\frac{\gamma}{c}(c-u)\cancel{t}\right)$$

$$c^2 = \gamma^2(c^2-u^2)$$

$$\gamma^2 = \frac{c^2}{c^2-u^2} = \frac{1}{1-u^2/c^2}$$

$$\gamma = \frac{1}{\sqrt{1-u^2/c^2}}$$ (the desired γ factor)

Lorentz Transformation (derivation)

Now we try to get an eq for t in terms of x' and t' by eliminating the x variable in the original two equations:

$$x' = \gamma(x - ut) \qquad x = \gamma(x' + ut')$$

$$x' = \gamma(\gamma(x' + ut') - ut)$$

$$x' = \gamma^2 x' + \gamma^2 ut' - \gamma ut$$

$$\gamma ut = \gamma^2 ut' + (\gamma^2 - 1)x'$$

$$t = \gamma t' + \frac{\gamma^2 - 1}{\gamma u} x' = \gamma\left(t' + \frac{\gamma^2 - 1}{\gamma^2 u} x'\right)$$

Simplifying the expression with γ's on the RHS,

$$\gamma = \frac{1}{\sqrt{1 - u^2/c^2}} = \frac{c}{\sqrt{c^2 - u^2}} \quad \rightarrow \quad \gamma^2 = \frac{c^2}{c^2 - u^2}$$

Lorentz Transformation (derivation)

More algebra…,

$$\gamma^2 - 1 = \frac{c^2}{c^2 - u^2} - 1 = \frac{c^2 - c^2 + u^2}{c^2 - u^2} = \frac{u^2}{c^2 - u^2}$$

$$\frac{\gamma^2 - 1}{\gamma^2 u} = \frac{u^{\cancel{2}}}{\cancel{c^2 - u^2}}\left(\frac{\cancel{c^2 - u^2}}{c^2}\right)\frac{1}{\cancel{u}} = \frac{u}{c^2}$$

$$\Rightarrow \quad t = \gamma\left(t' + \frac{\gamma^2 - 1}{\gamma^2 u}x'\right) = \gamma\left(t' + \frac{u}{c^2}x'\right)$$

(the desired Lorentz Transform in time)

Lorentz Transformation

$$S \to S' \begin{cases} x' = \gamma(x - ut) \\ y' = y \\ z' = z \\ t' = \gamma\left(t - \dfrac{u}{c^2}x\right) \end{cases} \qquad S' \to S \begin{cases} x = \gamma(x' + ut') \\ y = y' \\ z = z' \\ t = \gamma\left(t' + \dfrac{u}{c^2}x'\right) \end{cases}$$

Lorentz Transformation

Reduction back to Galilean Transformation in the regime: $u \ll c$.

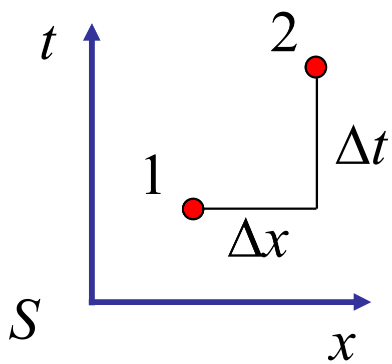
$$\text{For } \frac{u}{c} \ll 1 \quad \rightarrow \quad \gamma = \frac{1}{\sqrt{1-u^2/c^2}} \cong 1$$

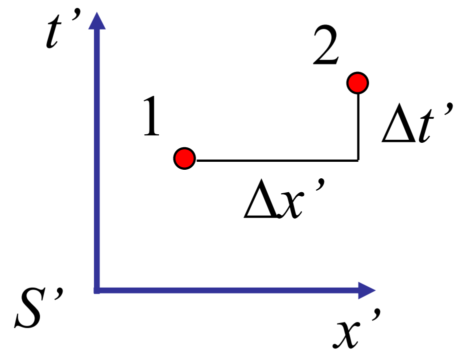
$$\Rightarrow \begin{cases} x' \cong 1\left(x-ut\right) = x-ut \\ t' = 1\left(t - \cancel{\dfrac{u}{c^2}x}\right) = t \end{cases} \qquad \text{(Galilean Transformation)}$$

Einstein Relativity is more general and it reduces to previous results (Galilean) in the $u \ll c$ limit.

Intervals between Two Events

We have defined Lorentz Transformation for an event *P*. Now, we want to extend it to a *space-time interval* (Δx, Δt) between two events:





Events 1 and 2 will have different coordinates for different observers *O* and *O*'.

$$\Delta x = x_2 - x_1,\ \Delta t = t_2 - t_1$$

$$\Delta x' = x'_2 - x'_1,\ \Delta t' = t'_2 - t'_1$$

In general, $\Delta x \neq \Delta x' \quad and$

$$\Delta t \neq \Delta t'$$

Intervals between Two Events

By direct substitution, we have the following transform for intervals between two events:

$$\begin{cases} \Delta x = \gamma\left(\Delta x' + u\Delta t'\right) \\ \Delta t = \gamma\left(\Delta t' + \dfrac{u}{c^2}\Delta x'\right) \end{cases}$$

or,

$$\begin{cases} dx = \gamma\left(dx' + u\,dt'\right) \\ dt = \gamma\left(dt' + \dfrac{u}{c^2}dx'\right) \end{cases}$$

for differential changes

Note: One can show that the combination $ds^2 = dx^2 - c^2dt^2 = ds'^2 = dx'^2 - c^2dt'^2$ is the same (*invariant*) for all inertial observation.

Time Dilation (Revisit)

In S'-frame, consider two clicks (1 & 2) of a clock stationary in S'.

- these two events occur at the *same* place ($x'_1 = x'_2$, $\Delta x' = 0$) but at different time ($t'_1 \neq t'_2$, $\Delta t' \neq 0$).

Now according to S-frame, $t_2 - t_1 = \Delta t = \gamma\left(\Delta t' + \cancel{\frac{u}{c^2}\Delta x'}\right) = \gamma\,\Delta t'$

Since the two clicks occurs *stationary* in S', $\Delta t'$ is the *proper time,* $\Delta t_0 = \Delta t'$.

$$\Delta t = \gamma\,\Delta t_0$$

So, the time interval measurement in S-frame is time dilated.

Simultaneity (Revisit)

In *S*'-frame, let consider two events 1 & 2 at two *different* locations: $x'_1 \neq x'_2,\ \Delta x' \neq 0$ happening at the same time so that

$$\Delta x' \neq 0,\ \Delta t' = 0$$

Then, in *S*-frame, these two events will be separated by the following time interval:

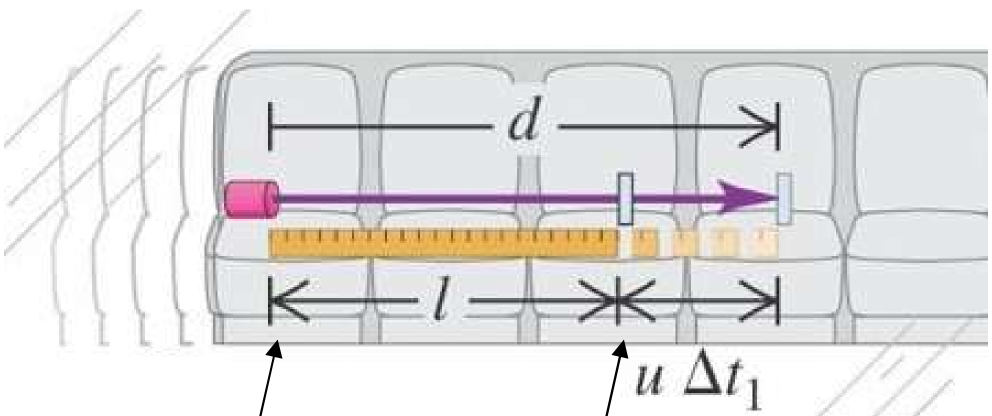
$$t_2 - t_1 = \Delta t = \gamma\left(\cancel{\Delta t'} + \frac{u}{c^2}\Delta x'\right) \Longrightarrow \Delta t = \gamma\frac{u}{c^2}\Delta x' \neq 0!$$

Thus, event 1 and 2 are *not* happening at the same time (*not* simultaneous) in *S*-frame !

Length Contraction (Revisit)

Consider a ruler at rest in *S'*-frame (train) so that its length between the two end points is given by $\Delta x' = L_0$ (the proper length).

How does the observer in *S*-frame measure the length of the moving ruler?



left end (at t=0)   right end (at t=0)

l was measured → by inferring the location of the right end of the ruler *at the same time* (at t=0) as the left end of the ruler.

Length Contraction (Revisit)

An observer in *S*-frame "measures" the two ends of this ruler *at the same time* according to his/her own clock ($\Delta t = 0$).

$$\Delta x' = \gamma\left(\Delta x \cancel{- u\Delta t}\right) = \gamma\,\Delta x$$

In *S*–frame, the measured length of the ruler is ($\Delta x = L$) and it is an *improper* length while $\Delta x'$ measured in *S*'-frame is proper. So we have,

$$\Delta x' = \gamma\,\Delta x$$

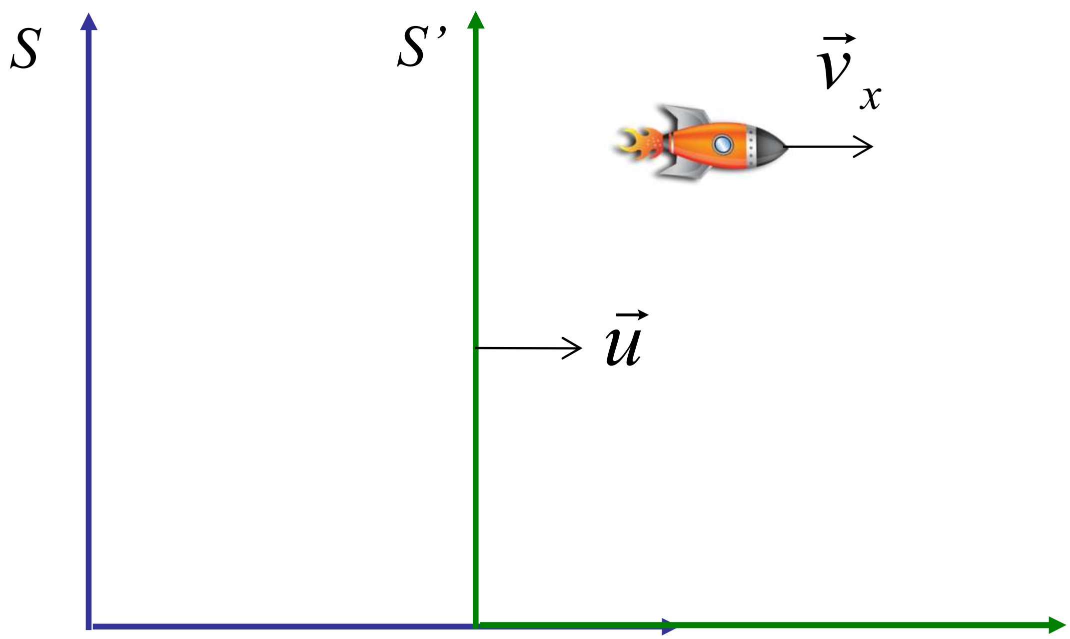
$$L_0 = \gamma\,L$$

This is the length contraction formula.

Lorentz Velocity Transformation

In *S*- frame, let say that we have an object moving in the *x*-direction with speed,

$$v_x = \frac{dx}{dt}$$



Note:
u → relative speed between *S* and *S*'
v_x → speed of object

Lorentz Velocity Transformation

Lorentz Transform gives:
$$\begin{cases} dx = \gamma\left(dx' + u\,dt'\right) \\ dt = \gamma\left(dt' + \dfrac{u}{c^2}dx'\right) \end{cases}$$

$$\Rightarrow \quad v_x = \frac{dx}{dt} = \frac{\gamma\left(dx' + u\,dt'\right)}{\gamma\left(dt' + \dfrac{u}{c^2}dx'\right)}$$

$$v_x = \frac{\cancel{\gamma}}{\cancel{\gamma}}\frac{dx' + u\,dt'}{dt' + \dfrac{u}{c^2}dx'}\left(\frac{1/dt'}{1/dt'}\right) = \frac{\dfrac{dx'}{dt'} + u}{1 + \dfrac{u}{c^2}\dfrac{dx'}{dt'}}$$

Lorentz Velocity Transformation

In *S'* - frame, the velocity of the object is defined as,

$$v'_x = \frac{dx'}{dt'}$$

This then gives,

$$v_x = \frac{\frac{dx'}{dt'} + u}{1 + \frac{u}{c^2}\frac{dx'}{dt'}} \Rightarrow \boxed{v_x = \frac{v'_x + u}{1 + \frac{u}{c^2}v'_x}}$$

(Lorentz Velocity Transformation)

Slow relative speed ($u << c$):

$$u/c << 1$$

$$\Rightarrow \quad v_x \cong \frac{v'_x + u}{1+0} = v'_x + u$$

(Galilean Velocity Transform)

Object moving at the speed of light:

$$v'_x = c$$

$$\Rightarrow \quad v_x = \frac{c+u}{1+\frac{u}{c^{\cancel{2}}}\cancel{c}} = \frac{c(1+u/c)}{1+u/c} = c$$

(c is the same in all frames)

Lorentz Velocity Transformation

If the object has velocity components in y and z directions: v'_y & v'_z, how would these components transform (u is in x direction only)?

$$v_y = \frac{dy}{dt} = \frac{dy'}{\gamma\left(dt' + \frac{u}{c^2}dx'\right)}\left(\frac{1/dt'}{1/dt'}\right) = \frac{\frac{dy'}{dt'}}{\gamma\left(1 + \frac{u}{c^2}\frac{dx'}{dt'}\right)}$$

$$v_y = \frac{v'_y}{\gamma\left(1 + \frac{u}{c^2}v'_x\right)}$$

Similarly for the z-component !

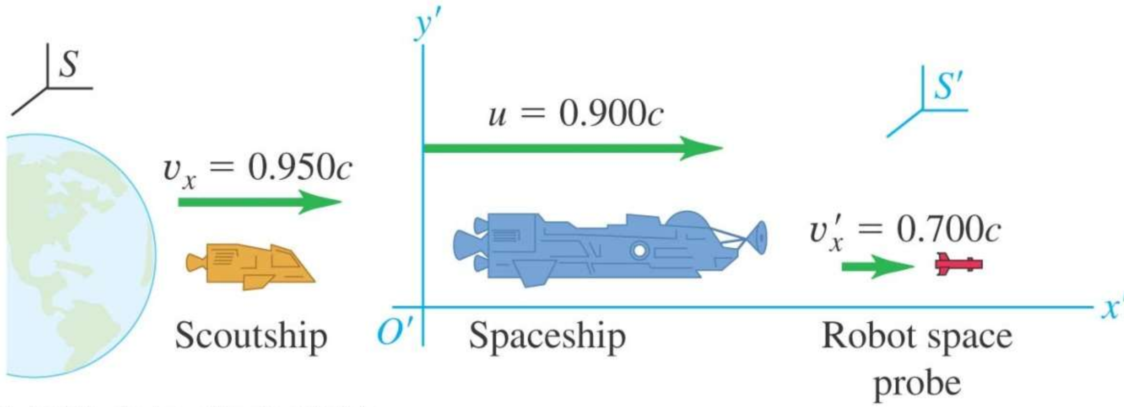
Lorentz Velocity Transform

From the principle of relativity, there should be no physical distinction for the two inertial observers in relative motion.

So the Lorentz Velocity Transform equation and its inverse transform should have the same form but with $u \leftrightarrow -u$ for the inverse transform of $\boldsymbol{v}$ in term of $\boldsymbol{v}'$.

$$\begin{cases} v_x = \dfrac{v'_x + u}{1 + \dfrac{u}{c^2} v'_x} \\ v_y = \dfrac{v'_y}{\gamma\left(1 + \dfrac{u}{c^2} v'_x\right)} \end{cases} \qquad \begin{cases} v'_x = \dfrac{v_x - u}{1 - \dfrac{u}{c^2} v_x} \\ v'_y = \dfrac{v_y}{\gamma\left(1 - \dfrac{u}{c^2} v_x\right)} \end{cases}$$

Example 37.7



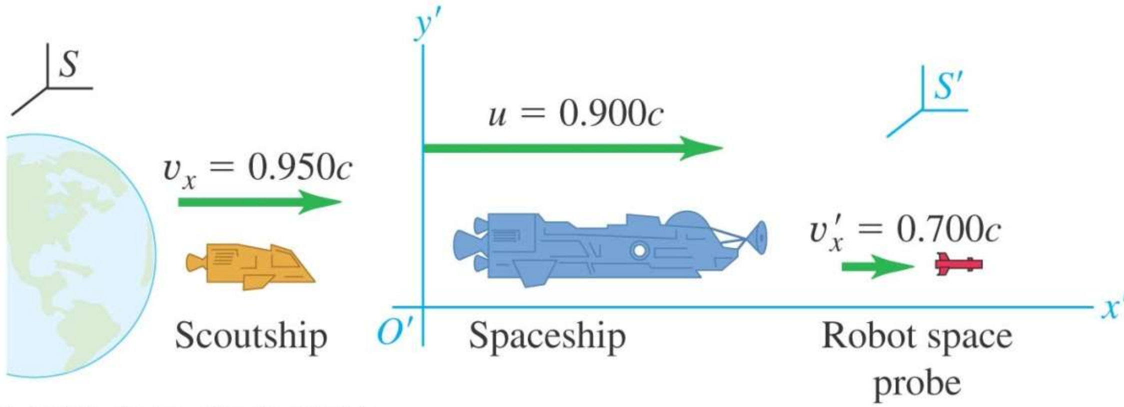
a) What is the probe's velocity relative to earth?
b) What is the scoutship's velocity relative to spaceship?

Setup: Two frames: S→Earth, S'→ Spaceship, $u = +\,0.900c$

a) In S'-frame, the probe moves at $v_x'(probe) = 0.700c$

$$v_x = \frac{v'_x + u}{1 + uv'_x / c^2} = \frac{0.700c + 0.900c}{1 + (0.700)(0.900)} = 0.982c$$

Example 37.7



a) The scoutship's speed is given in S-frame (with respect to Earth),

$$v_x\left(scout\right) = +0.950c$$

The scoutship's speed with respect *S*'-frame is given by the *inverse* transform,

$$v'_x = \frac{v_x - u}{1 - uv_x/c^2} = \frac{0.950c - 0.900c}{1 - (0.950)(0.900)} = 0.345c$$

Relativistic Momentum & Energy

As we have seen, time intervals, length intervals, and velocity change according to the Lorentz Transformation depending on the observer's frame of reference.

$$(x,t) \quad \leftrightarrow \quad (x',t') \quad \text{Lorentz Transformation}$$

Other *dynamical quantities* (such as momentum, energy, etc.) must also be appropriately expressed so that the laws of physics satisfy the following conditions:

- Satisfy the two postulates of Special Relativity:
  - Laws of physics (e.g., conservation of momentum, conservation of energy, Newton's laws) apply equally to all inertial observers.
  - Speed of light in vacuum same for all inertial observers
- The modified relativistic dynamical quantities should reduce to the classical ones for $u \ll c$.

Relativistic Momentum & Energy

Relativistic Momentum:

$$\vec{\mathbf{P}} = \gamma m\vec{\mathbf{v}} = \frac{m\vec{\mathbf{v}}}{\sqrt{1-v^2/c^2}}$$

Momentum of a particle moving with velocity $\vec{\mathbf{v}}$ as measured in the lab frame (S-frame).

Relativistic Energy:

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1-v^2/c^2}}$$

Total energy of a particle moving with velocity $\vec{\mathbf{v}}$ as measured in lab frame (S-frame).

Relativistic Momentum & Energy

All laws of physics remain valid in all inertial reference frames means.

Conservation Laws for relativistic $\vec{\mathbf{P}}$ & E must remain the same!

→ Experimentally, it has been shown repeatedly that it is ($\gamma m\vec{\mathbf{v}}$ and γmc^2) rather than their classical counterparts that are conserved in high energy collisions!

→ For $v << c$, $\gamma = \dfrac{1}{\sqrt{1-v^2/c^2}} \cong 1$ So, $\vec{\mathbf{P}} = \gamma m\vec{\mathbf{v}} \rightarrow m\vec{\mathbf{v}}$

So that for non-relativistic speeds, relativistic $\vec{\mathbf{P}}$ reduces to classical $\vec{\mathbf{P}}$.

Relativistic Force

$$\vec{\mathbf{F}} = \frac{d\vec{\mathbf{P}}}{dt}$$

Newton's 2nd Law: same form as in the classical case but with relativistic momentum $\vec{\mathbf{P}}$.

Note: If $\vec{\mathbf{F}} = 0$ (no external force), momentum $\vec{\mathbf{P}}$ as expected will be *conserved* in *both* relativistic & classical regimes !

Let use the relativistic force to consider the **work-energy theorem**:

Work done by F to accelerate a relativistic particle from v_i to v_f ⇒

$$W = \int_i^f F\,dx = \int_i^f \frac{dP}{dt}\,dx$$

(for simplicity, F is in x-direction only)

Relativistic Work & Energy

$$\int_i^f \frac{dP}{dt}\,dx = \int_i^f \frac{dP}{\cancel{dt}}\,v\,\cancel{dt} \quad \Rightarrow \quad W = \int_i^f v\,dP$$

Now, $d(Pv) = vdP + Pdv \quad \rightarrow \quad vdP = d(Pv) - Pdv$

Substituting this into the above equation for W:

$$W = \int_i^f v\,dP = \underbrace{\int_i^f d(Pv)}_{1^{st}\ \text{term}} - \underbrace{\int_i^f P\,dv}_{2^{nd}\ \text{term}}$$

Relativistic Work & Energy

$1st$ term $= Pv\Big|_i^f = (\gamma m v_f)v_f - (\gamma m v_i)v_i$

For simplicity, choose $v_i = 0$ and $v_f = v$

Then, the 1st term becomes: $\int_i^f d(Pv) = \gamma m v^2 = \dfrac{mv^2}{\sqrt{1 - v^2/c^2}}$

Now, let consider the 2nd term,

$$\int_i^f P\,dv = \int_i^f \frac{mv}{\sqrt{1 - v^2/c^2}}\,dv$$

We can integrate this by a simple change of variable.

Relativistic Work & Energy

Let $s = \left(1 - v^2/c^2\right)^{1/2}$

$$ds = \frac{1}{2}\left(1 - v^2/c^2\right)^{-1/2}\left(-2v/c^2\right)dv$$

$$ds = -\frac{v/c^2}{\sqrt{1 - v^2/c^2}}\,dv$$

So, $\int_i^f P\,dv = \int_i^f \frac{mv}{\sqrt{1 - v^2/c^2}}\,dv = -mc^2 \int_i^f \frac{-v/c^2}{\sqrt{1 - v^2/c^2}}\,dv = -mc^2 \int_i^f ds$

$$= -mc^2\left(s_f - s_i\right)$$

$$= -mc^2\sqrt{1 - v^2/c^2} + mc^2\sqrt{1 - 0^2/c^2}$$

$$= -mc^2\sqrt{1 - v^2/c^2} + mc^2$$

Relativistic Work & Energy

Putting everything together,

$$W = \frac{mv^2}{\sqrt{1-v^2/c^2}} + mc^2\sqrt{1-v^2/c^2} - mc^2$$

$$= \frac{mv^2}{\sqrt{1-v^2/c^2}} + mc^2\sqrt{1-v^2/c^2}\,\frac{\sqrt{1-v^2/c^2}}{\sqrt{1-v^2/c^2}} - mc^2$$

$$= \frac{mv^2}{\sqrt{1-v^2/c^2}} + \frac{mc^2}{\sqrt{1-v^2/c^2}}\left(1-v^2/c^2\right) - mc^2$$

$$= \frac{\cancel{mv^2} + mc^2 \cancel{-mv^2}}{\sqrt{1-v^2/c^2}} - mc^2 = \frac{mc^2}{\sqrt{1-v^2/c^2}} - mc^2$$

Relativistic Work & Energy

Applying the **work-energy theorem**, this amount of work done to accelerate the particle from 0 to v should equal to the change in KE.

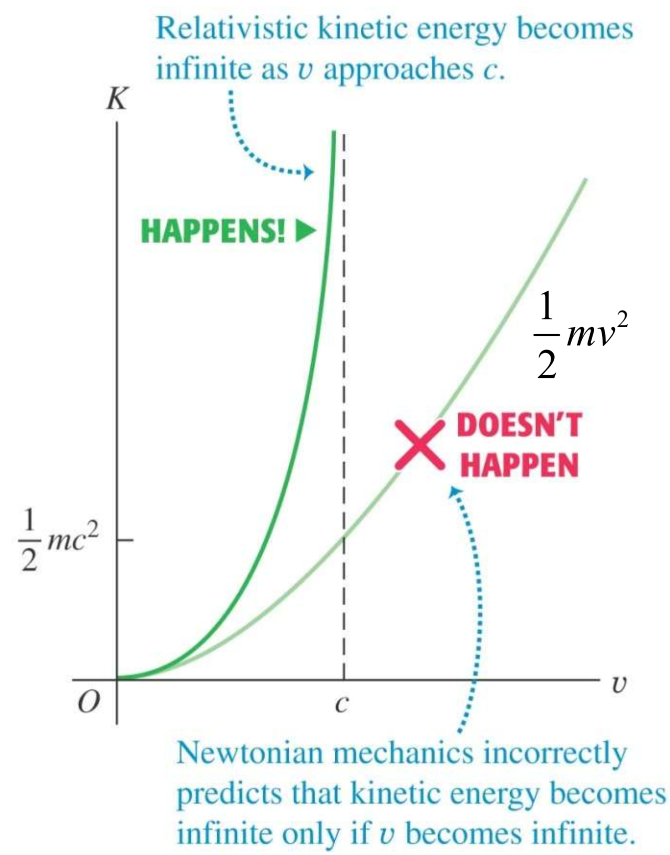
(Laws of physics should be unchanged in relativity !)

Since $v_i = 0, \quad KE_i = 0$

$v_f = v, \quad KE_f = KE$

$$W = \boxed{KE = \frac{mc^2}{\sqrt{1 - v^2/c^2}} - mc^2 = \gamma mc^2 - mc^2}$$

Relativistic Kinetic Energy of a particle moving with $\vec{\mathbf{v}}$ as measured in the lab frame.



Relativistic KE → Classical KE

Slow moving particle regime $v << c$:

Using binomial theorem, $\left(1-\frac{v^2}{c^2}\right)^{-1/2} = 1+\left(-\frac{1}{2}\right)\left(-\frac{v^2}{c^2}\right)+O\left(\left(\frac{v^2}{c^2}\right)^2\right) \cong 1+\frac{1}{2}\frac{v^2}{c^2}$

Substituting this into the equation for Relativistic KE,

$$KE = \left(1-\frac{v^2}{c^2}\right)^{-1/2} mc^2 - mc^2 = \left[\left(1-\frac{v^2}{c^2}\right)^{-1/2} - 1\right] mc^2$$

$$\cong \left[1+\frac{1}{2}\frac{v^2}{c^2} - 1\right] mc^2$$

$$= \frac{1}{2} mv^2$$

This is the classical result for $v << c$.