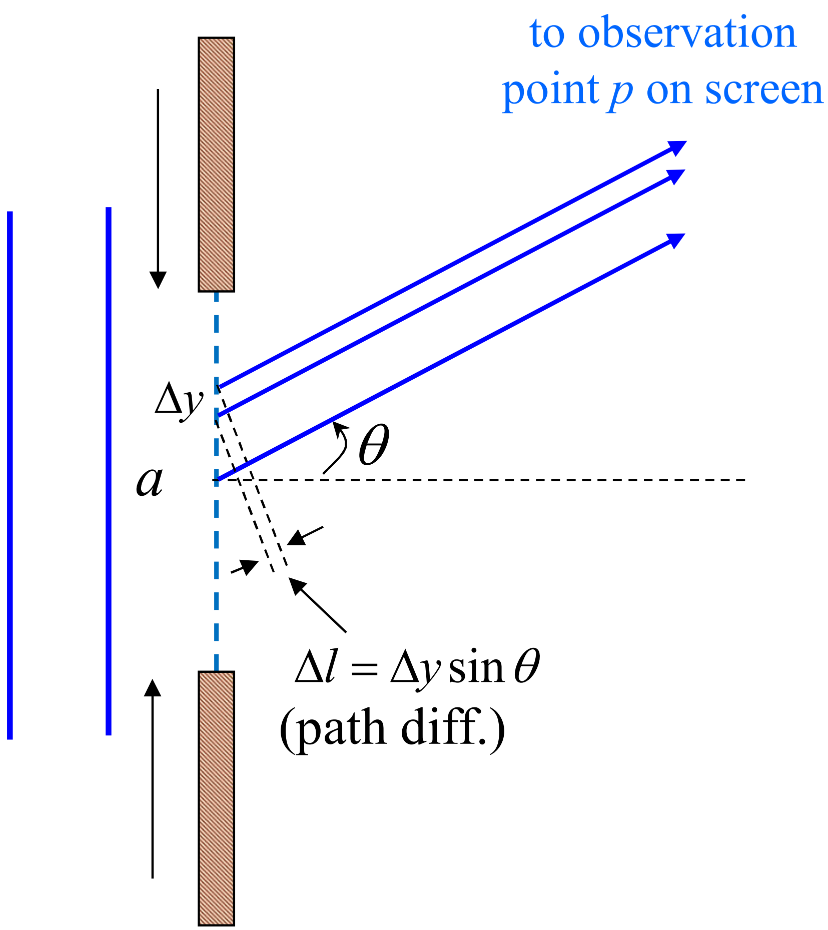
Intensity in Single-Slit Pattern



Divide the wave front at the slit into N (large #) smaller strips/wavelets.

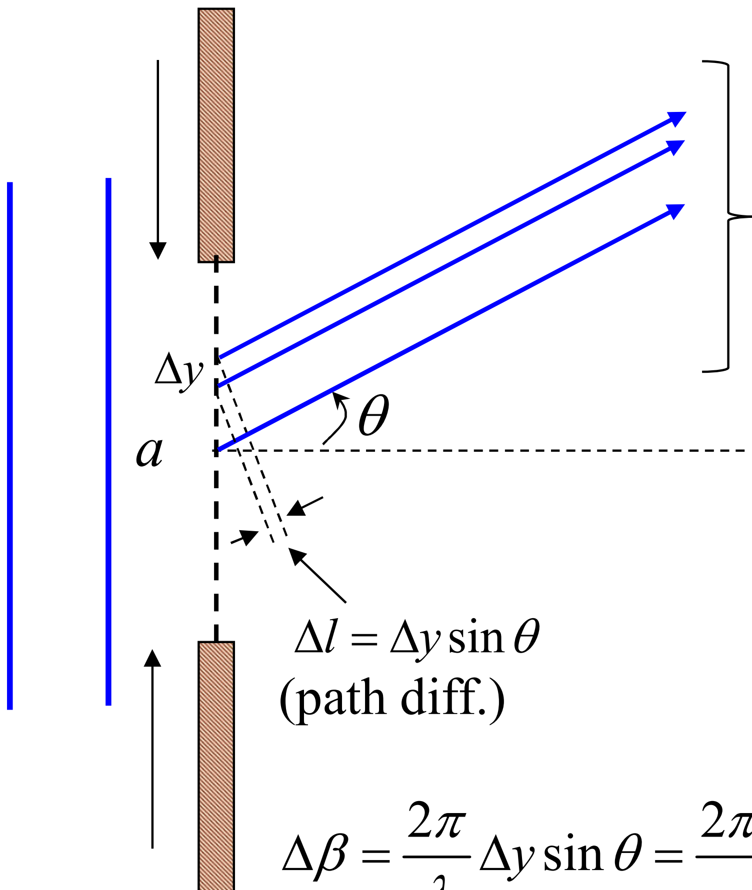
$\Delta y = a / N \quad \rightarrow$ size of each small strip

$\Delta l = \Delta y \sin\theta \rightarrow$ small path diff. bet adj. strips

If E_0 is the magnitude of the incoming wave, then the E-field from each wavelet will have a magnitude of

$$\Delta E = E_0 / N$$

Intensity in Single-Slit Pattern (Phasors)



Consider the electric field from each wavelet as a phasor $\Delta\vec{\mathbf{E}}$, the resultant electric field E_p at p can be calculated as the phasor sum of all the $\Delta\vec{\mathbf{E}}\,'s$.

$$\Delta\vec{\mathbf{E}}_1, \Delta\vec{\mathbf{E}}_2, \cdots, \Delta\vec{\mathbf{E}}_N$$

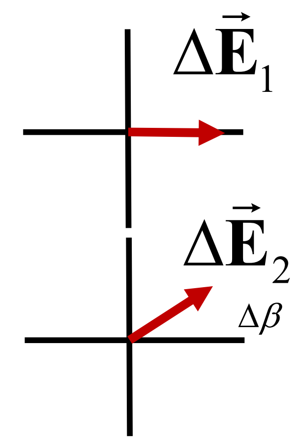
$$\Delta\beta = \frac{2\pi}{\lambda}\Delta y \sin\theta = \frac{2\pi a \sin\theta}{N\lambda}$$

$$\Delta E_1 = \frac{E_0}{N}\cos(\omega t) \longrightarrow$$

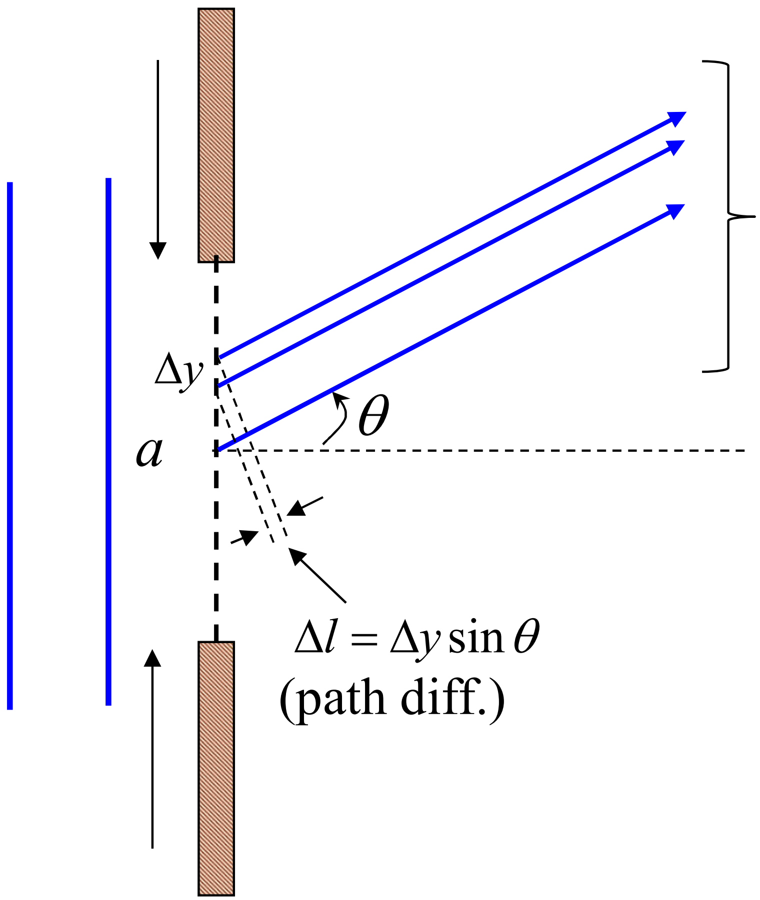
$$\Delta E_2 = \frac{E_0}{N}\cos(\omega t + \Delta\beta) \longrightarrow$$

$$\vdots$$

$$\Delta E_N = \frac{E_0}{N}\cos(\omega t + (N-1)\Delta\beta)$$



Intensity in Single-Slit Pattern (Phasors)



$$\Delta\vec{\mathbf{E}}_1, \Delta\vec{\mathbf{E}}_2, \cdots, \Delta\vec{\mathbf{E}}_N$$

For each pair of adjacent phasors, there is a path difference Δl

and this path difference induces a phase difference $\Delta\beta$ between adjacent phasors.

$$\frac{\Delta\beta}{2\pi} = \frac{\Delta l}{\lambda} \quad \rightarrow \quad \Delta\beta = \frac{2\pi}{\lambda}\Delta l = \frac{2\pi}{\lambda}\Delta y \sin\theta$$

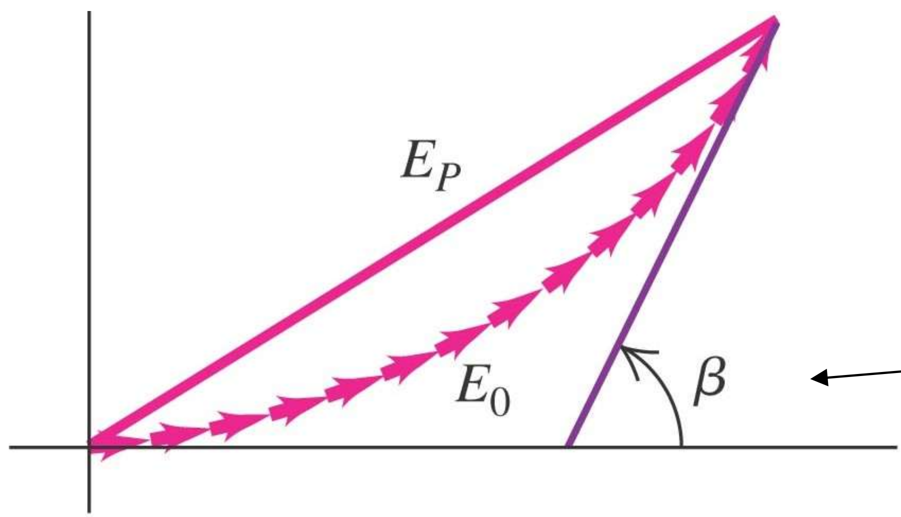
with $\Delta y = \dfrac{a}{N}$, we then have $\Delta\beta = \dfrac{2\pi a \sin\theta}{N\lambda}$

Phase Difference from Path Difference

Considering the phasor sum of all N phasors, the *total* phase difference β is,

$$\beta = N\Delta\beta = N\left(\frac{2\pi}{\lambda}\Delta y \sin\theta\right)$$

$$\beta = \frac{2\pi}{\lambda}(N\Delta y)\sin\theta = \frac{2\pi}{\lambda}a\sin\theta$$

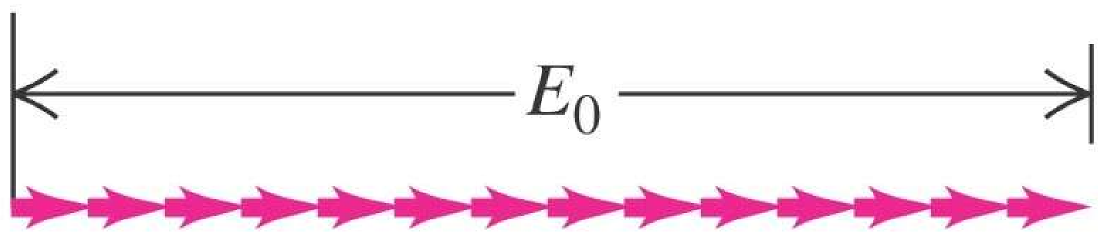


Note, the total phase difference β is again a function of the angular location θ.

NOTE: Similar to ϕ earlier, β has to be in radian!

Summing Phasors to Calculate E_p

Central Maximum ($\Delta\beta = 0$, $\beta = 0$, straight ahead):

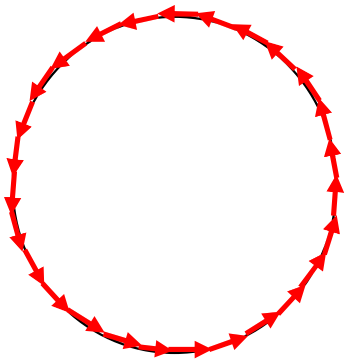


All phasors are *in phase.*

$$E_P = N\Delta E = E_0$$

Summing Phasors to Calculate E_p

First Order Minimum ($\beta = N\Delta\beta = 2\pi$):



1st minimum condition when last phasor's tip matches up *exactly* with the first phasor's end.

$$E_P = 0$$

Note: $\beta = \frac{2\pi}{\lambda} a \sin\theta = 2\pi \quad \rightarrow \quad a \sin\theta = \lambda$

same condition as previously derived.

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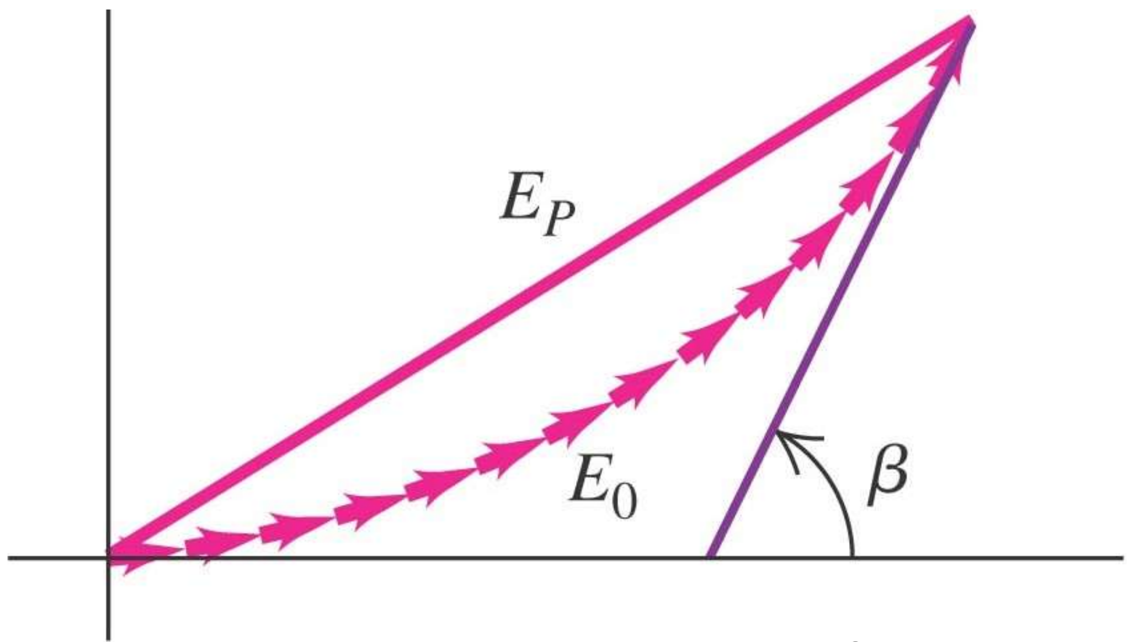
Summing Phasors to Calculate E_p

Slightly away from Central Maximum $(\Delta\beta > 0, \beta > 0)$:



β is the phase diff. between the first and the last phasors.

$$E_P < E_0$$

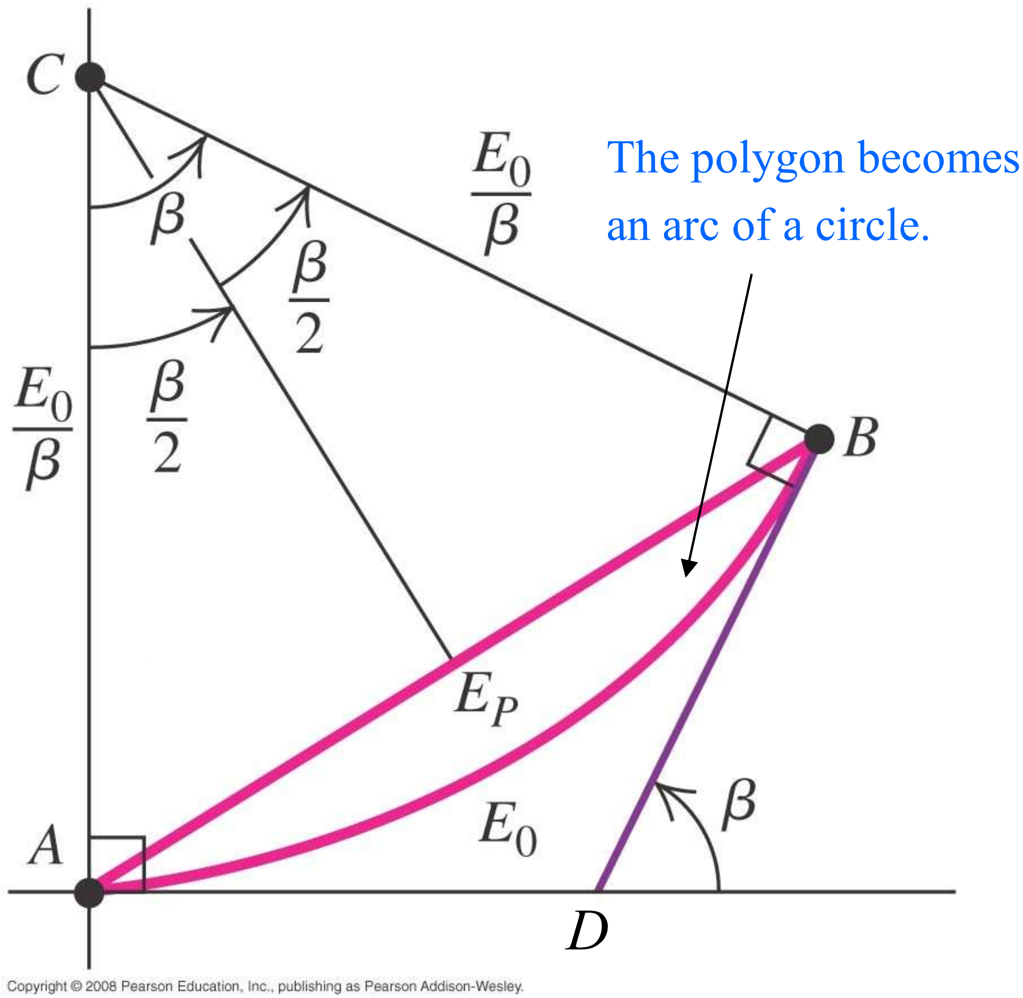
$$\beta = \frac{2\pi}{\lambda} a \sin\theta$$

$D_{mathematica}$

http://demonstrations.wolfram.com/RepresentationOfDiffractionByVectorSummation/

Intensity in Single-Slit Pattern

For $N \to \infty, \Delta y \to dy$ we can find an expression of the intensity I in terms of $\theta\,(\beta)$.



- C is the center of the arc
- angle A and B are right angles
- interior angle at D is 180^{o} - β

$$\angle C = \beta$$

For the circular section ACB,

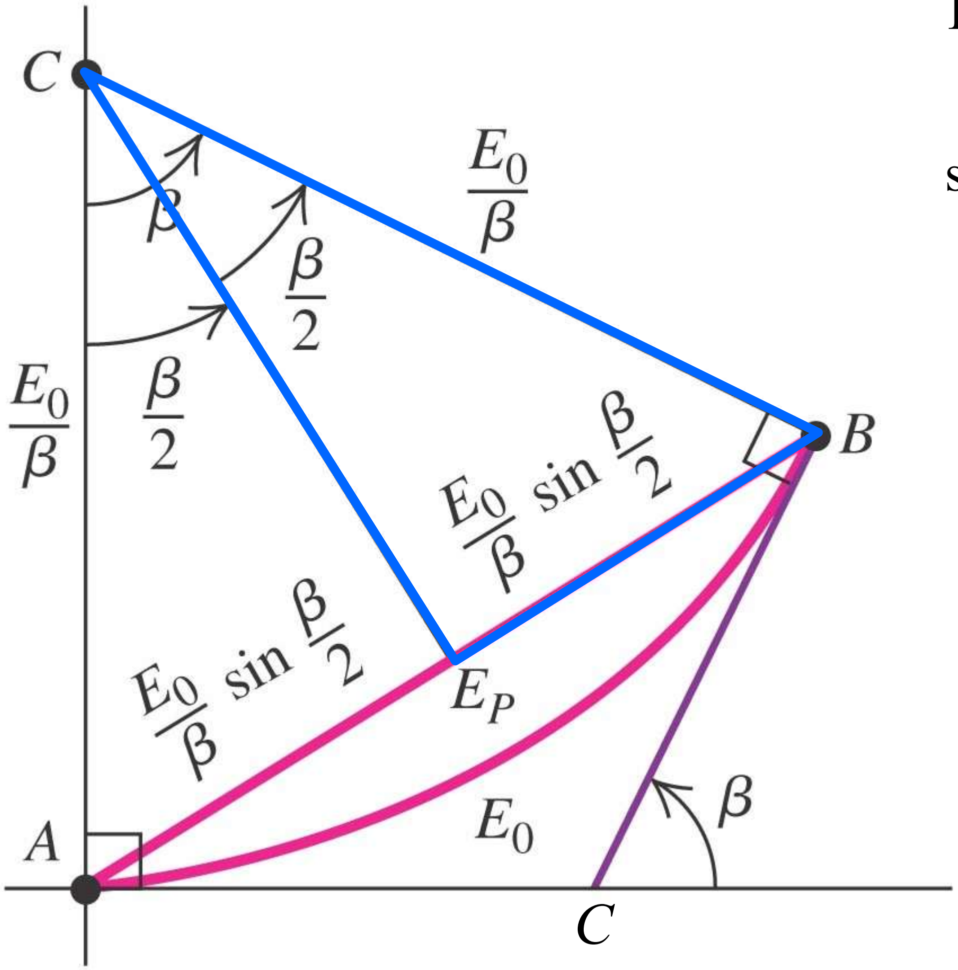
$$radius \times angle = arc\_lenght$$

$$radius \times \ \beta = E_0$$

(β has to be in radian)

$$radius = \frac{E_0}{\beta}$$

Intensity in Single-Slit Pattern



Lastly, from the blue right triangle,

$$\sin\frac{\beta}{2} = \frac{\overline{AB}/2}{E_0/\beta} \quad \rightarrow \quad \overline{AB} = E_p = E_0\frac{\sin(\beta/2)}{\beta/2}$$

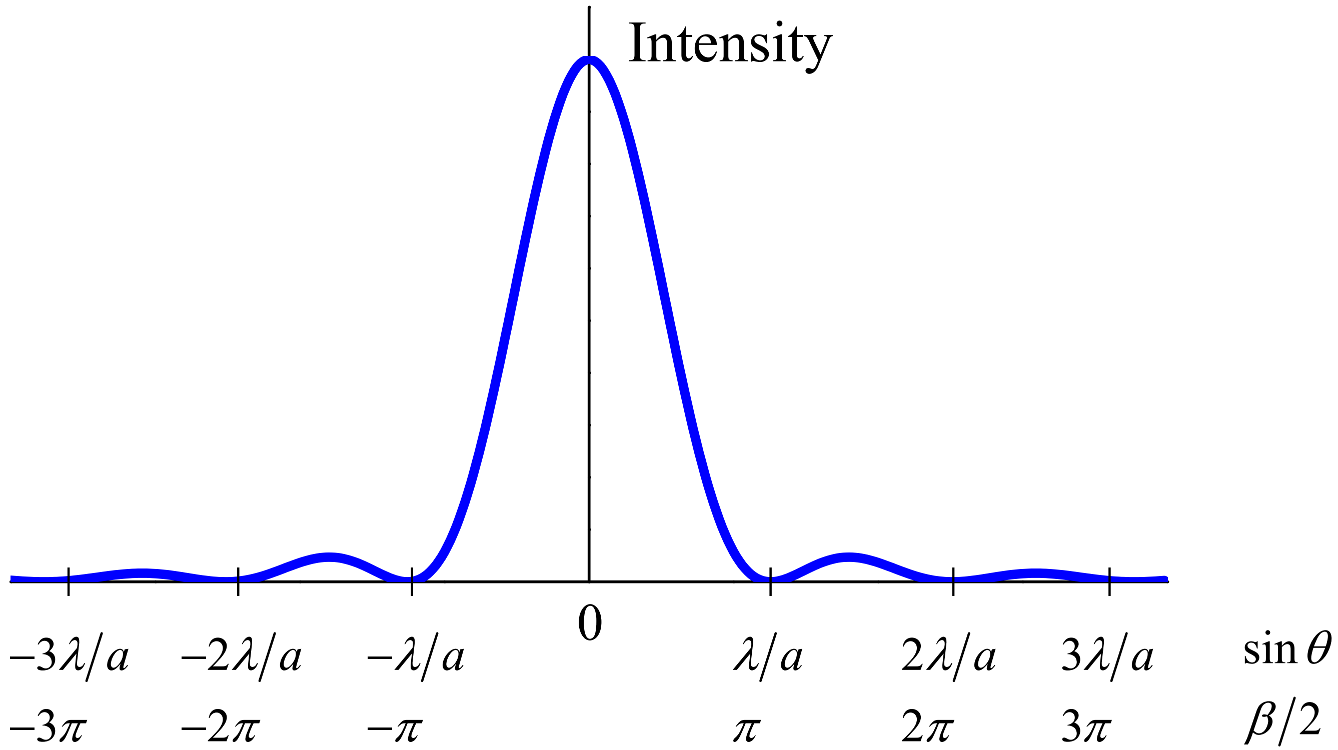
With I proportional to E_P^2,

$$I = I_0\left[\frac{\sin(\beta/2)}{\beta/2}\right]^2$$

Intensity in Single-Slit Pattern

Then, lastly with $\beta = \dfrac{2\pi}{\lambda} a \sin\theta$, the intensity of the pattern as a function of θ is,

$$I = I_0 \left[\frac{\sin\left(\pi a \sin\theta / \lambda\right)}{\pi a \sin\theta / \lambda} \right]^2$$



Locating Mins & Maxs

Minimum:

requires that $I = I_0\left[\dfrac{\sin(\beta/2)}{\beta/2}\right]^2 = 0,$ where $\dfrac{\beta}{2} = \dfrac{\pi}{\lambda} a\sin\theta$

$$\Rightarrow \quad \frac{\beta}{2} = \frac{\pi}{\lambda} a\sin\theta = m\pi, \quad m = \cancel{0}, \pm 1, \pm 2, \cdots$$

$$\Rightarrow \quad a\sin\theta = m\lambda$$

same condition as previously derived !

Note that $\beta/2 = 0$ ($m = 0$) is *not* a solution for a minimum !

In fact, $\dfrac{\beta}{2} = \dfrac{\pi}{\lambda} a\sin\theta$ or $\theta = 0$ is the central maximum with $\lim\limits_{\beta\to 0} \dfrac{\sin(\beta/2)}{\beta/2} = 1$

Locating Mins & Maxs

Maximum:

For single-slit Fraunhofer diffraction patterns, maximum occur *near*

$$\frac{\beta}{2} = \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \cdots \quad \text{but not exactly !}$$

To find the maximum exactly, we need to find the *extremum* for the intensity function, i.e.,

$$\frac{dI(x)}{dx} = \frac{d}{dx}\left(\frac{\sin x}{x}\right)^2 = 0 \qquad \text{where} \quad x = \frac{\beta}{2} = \frac{\pi}{\lambda} a \sin\theta$$

$$\Rightarrow \quad \frac{d}{dx}\left(\frac{\sin x}{x}\right)^2 = 2\left(\frac{\sin x}{x}\right)\left(\frac{\cos x}{x} - \frac{\sin x}{x^2}\right) = 0$$

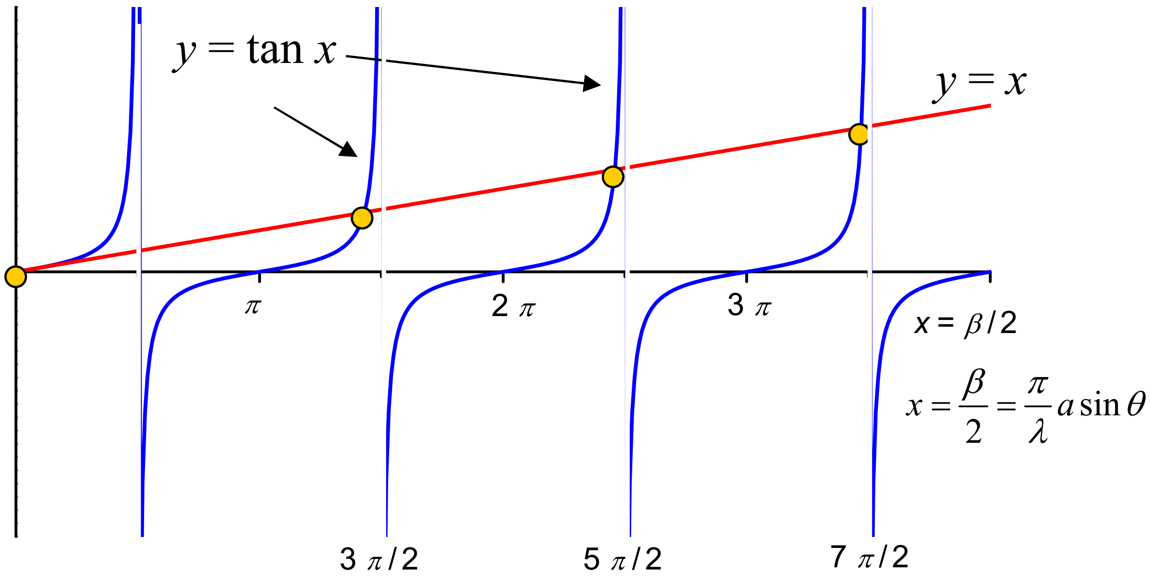
Locating Mins & Maxs

There are two solutions:

1. $$\frac{\sin x}{x} = 0 \quad \Rightarrow \quad x = m\pi \rightarrow \frac{\pi a \sin\theta}{\lambda} = m\pi \rightarrow a\sin\theta = m\lambda$$

(same *minimum* condition)

2. $$\frac{\cos x}{x} - \frac{\sin x}{x^2} = 0 \quad \rightarrow \quad x = \tan x$$



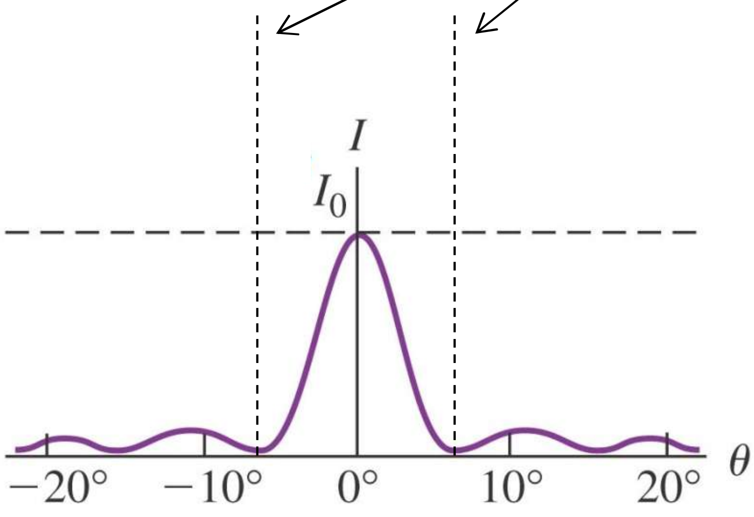
- intercepts are solutions to $\tan x = x$.

They gives the locations of the *maximum* of the intensity function.

Width of the Single-Slit Pattern

One can estimate the width of central max using the locations of 1st min on both left and right sides of the central max:

$$\sin\theta_1 = \pm\frac{\lambda}{a}$$



Width of the Single-Slit Pattern

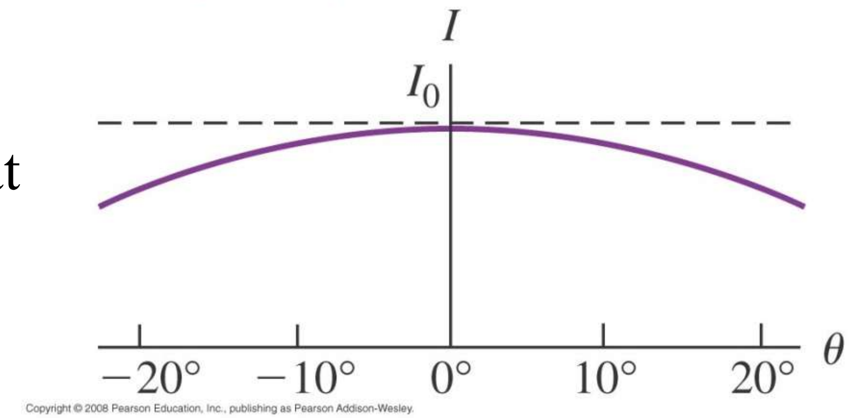
One can estimate the width of central max using the locations of 1st min on both left and right sides of the central max:

$$\sin\theta_1 = \pm\frac{\lambda}{a}$$

So, if $a \leq \lambda$, only one broad maximum is visible !

1st min at

-90°



1st min at

90°

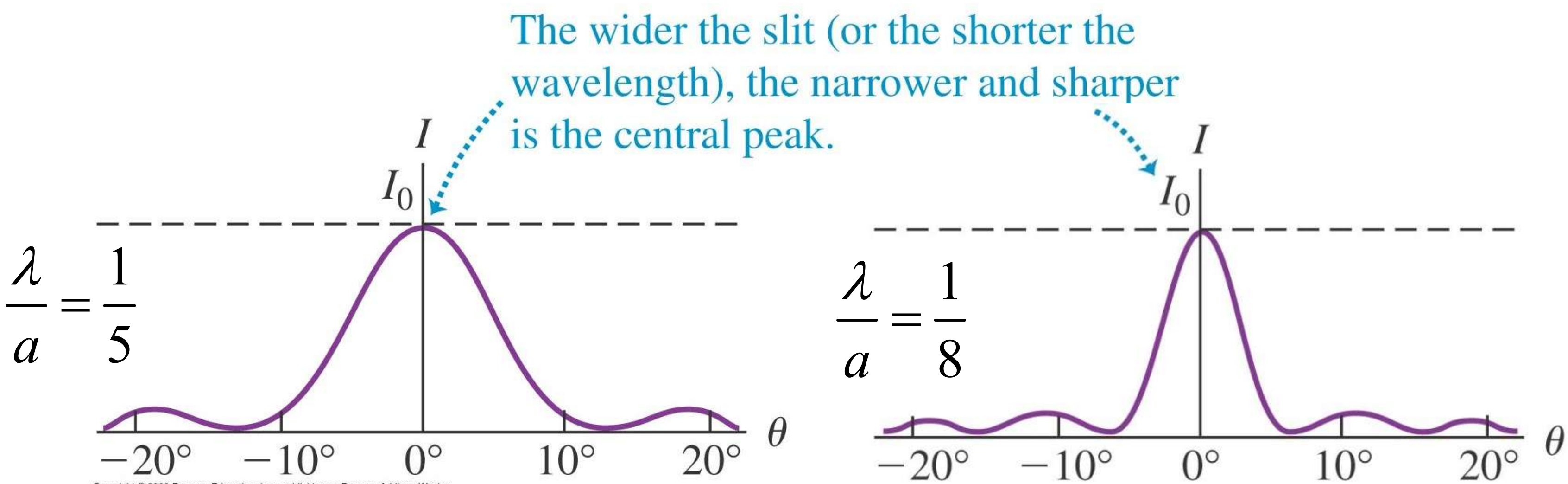
Width of the Single-Slit Pattern

One can estimate the width of central max using the locations of 1st min on both left and right sides of the central max:

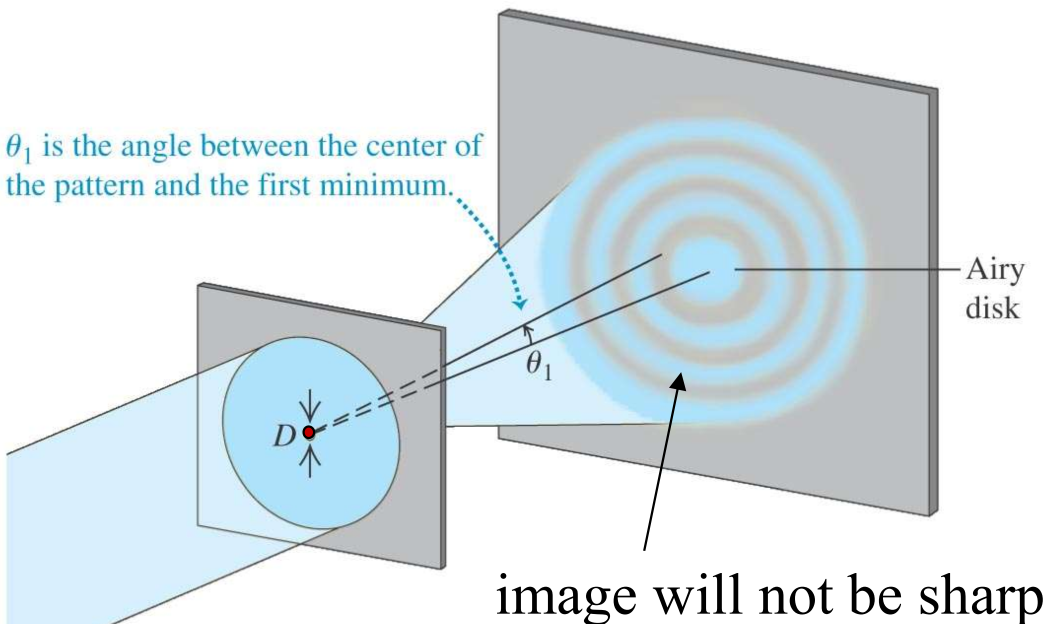
$$\sin\theta_1 = \pm\frac{\lambda}{a}$$

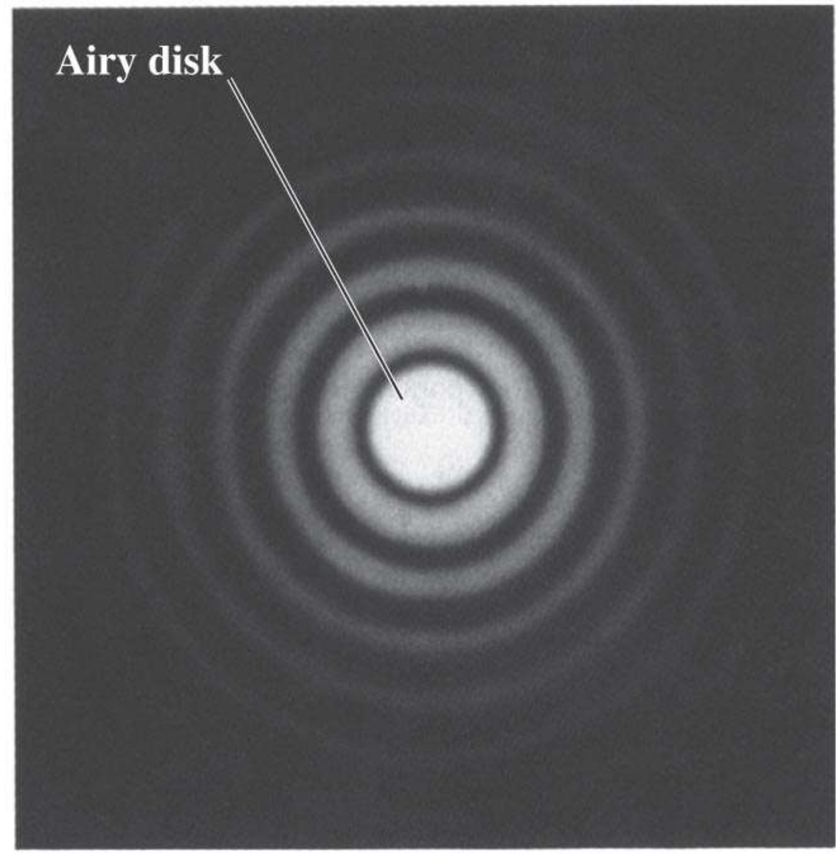
On the other hand, if $\lambda < a$ and as $\left|\frac{\lambda}{a}\right| \downarrow$

$\Rightarrow$ 1st min moves closer (peak sharper)!



Resolving Power for Circular Apertures

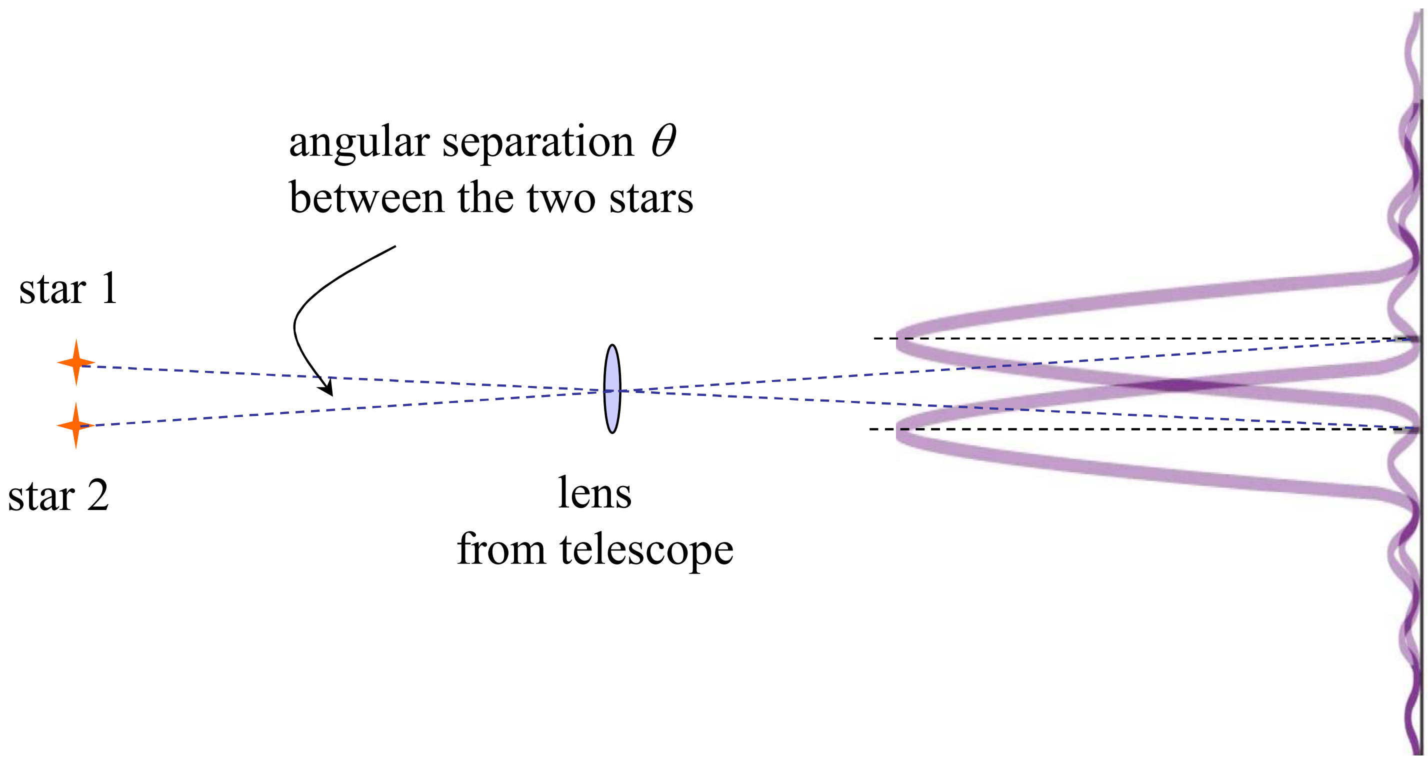




Because of *diffraction*, light spreads out after passing thru circular apertures → this imposes *resolution limits* to commonly used optical instruments, such as microscopes and telescopes.

Resolving Power for Circular Apertures

Consider two *non-coherent* point sources (so that they don't interfere), i.e. two distant stars,



We will observe two diffraction patterns on top of each others.

Resolving Power for Circular Apertures

The overlap of the two diffraction pattern might prevent one from discerning the two sources of light.

A workable criterion is called the **Rayleigh's Criterion** which is similar in spirit to our discussion for the resolving power for the diffraction grating:

> The two diffraction pattern can be resolvable if the central max from one pattern is at least as far as the 1^{st} min of the other image.

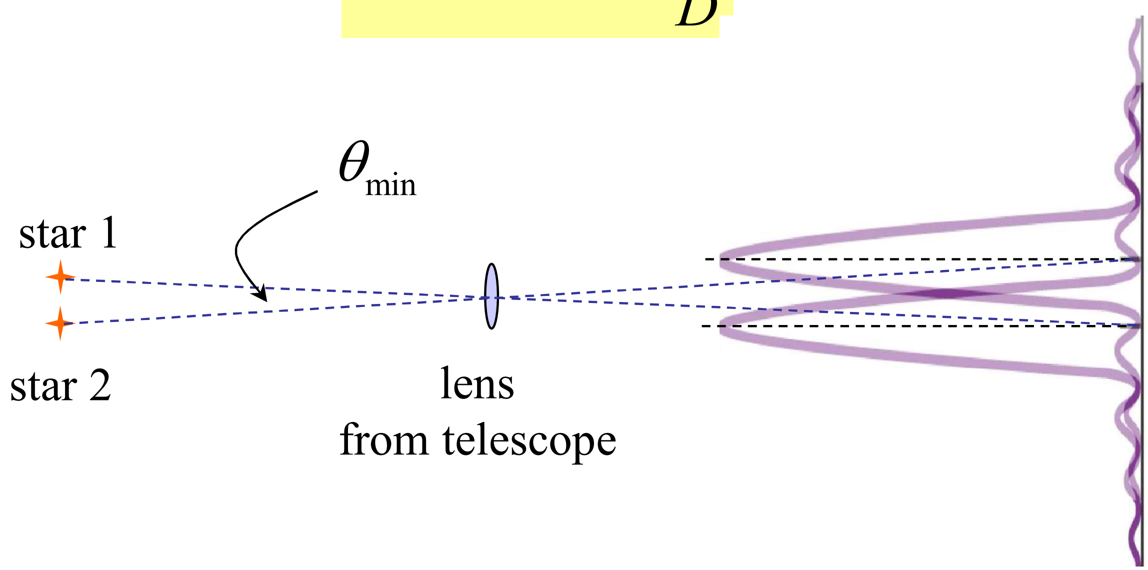
For circular aperture with diameter *D*, the angular location of the its 1^{st} order diffraction minimum is:

$$\sin\theta_1 = 1.22\frac{\lambda}{D} \qquad \text{("1.22" is a geometric factor)}$$

Resolving Power for Circular Apertures

The **Limit of Resolution** for a circular aperture is defined as the smallest angular separation between two light sources that can be resolved according to the *Rayleigh's Criterion* and it is given by:

$$\sin\theta_{\min} = 1.22\frac{\lambda}{D}$$



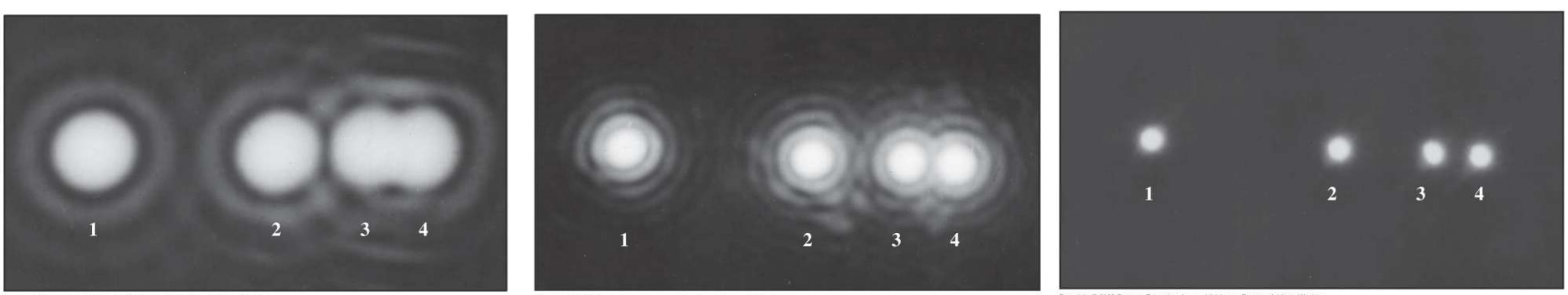
Resolving Power for Circular Apertures

An optical device such as a telescope or microscope will have a *high* **Resolving Power** if it has a *small* **Limit of Resolution** (θ_{min} small) so that nearby objects with a small angular separation can be resolved.

This gives the following ways to increase the Resolving Power:

- increase the diameter D → use a bigger len/mirror in telescope
- decrease the wavelength λ → use a shorter wavelength of light in chip production

(a) Small aperture (b) Medium aperture (c) Large aperture



Example 36.6: Resolving Power of a Camera Lens

Given:
f=50 mm
f-number of f/2
object distance 9.0m
wavelength = 500nm

What is the minimum distance between two points on the faraway object that one can resolve?

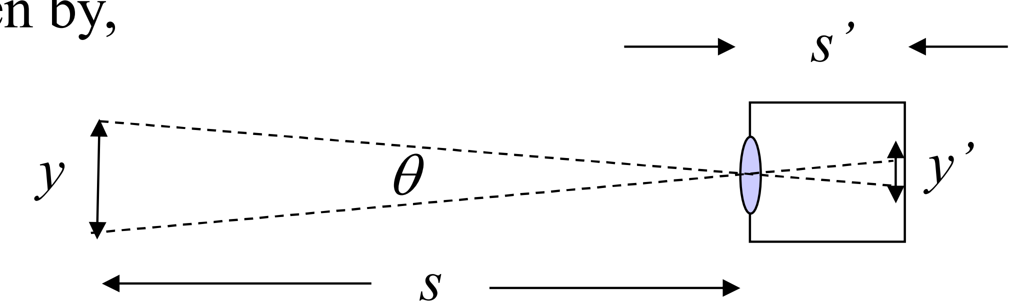
f-number = f/D → D = f/f-number = 50mm/2 =25 mm

Rayleigh's Criterion gives:

$$\theta_{\min} \cong \sin\theta_{\min} = 1.22\frac{\lambda}{D} = 1.22\frac{500\times10^{-9}\,m}{25\times10^{-3}\,m} = 2.4\times10^{-5}\,rad$$

Example 36.6: Resolving Power of a Camera Lens

For a simple lens, we know that the angular separation of two points on the object is given by,



$$\theta \cong \left|\frac{y}{s}\right| = \left|\frac{y'}{s'}\right|$$

y → separation of object points
y' → separation of the corresponding image points
s → object distance
s' → image distance

Example 36.6: Resolving Power of a Camera Lens

Applying the minimum condition, we have,

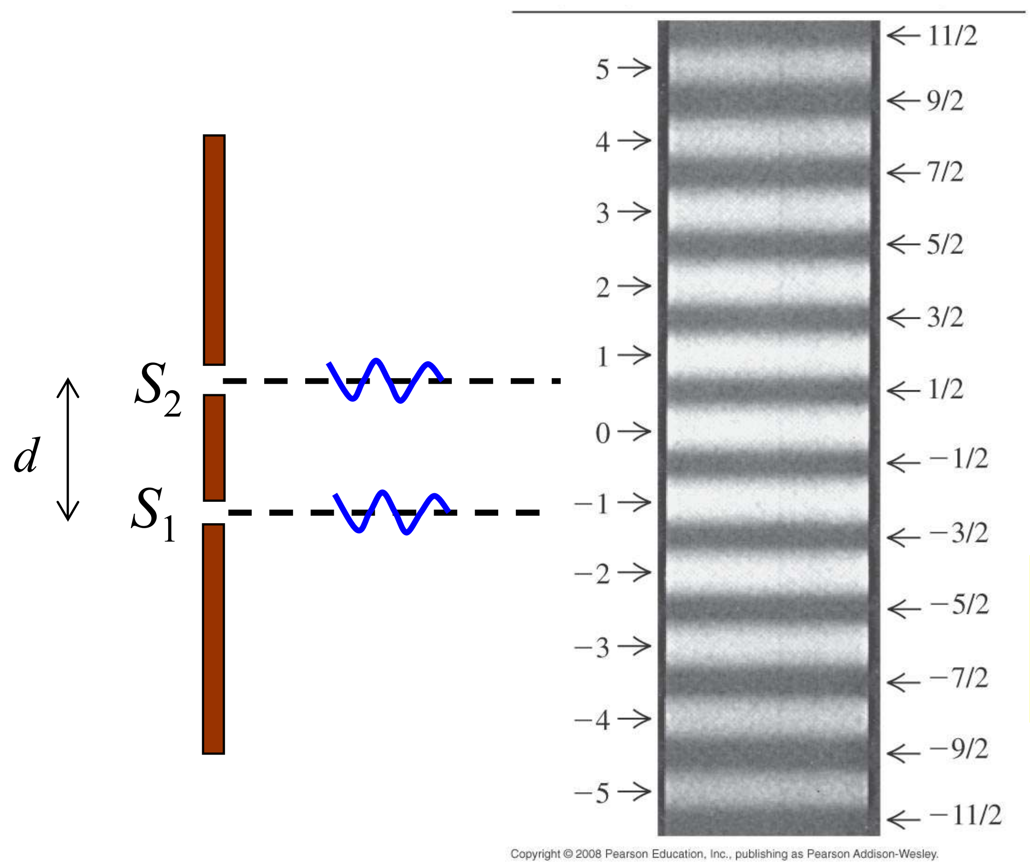
$$2.4\times10^{-5}\,rad = \frac{y}{s} \quad \rightarrow \quad y = 9.0m(2.4\times10^{-5}\,rad) = 0.22mm \qquad \text{(on object)}$$

On the camera film, the image separation will be approximately,

$$2.4\times10^{-5}\,rad = \frac{y'}{s'} \quad \rightarrow \quad y' = 50mm(2.4\times10^{-5}\,rad) = 1.2\times10^{-3}\,mm \qquad \text{(on film)}$$

$s' \sim f$ if image is focused on the film

Double-Slit Interference Pattern (w/o diffraction)



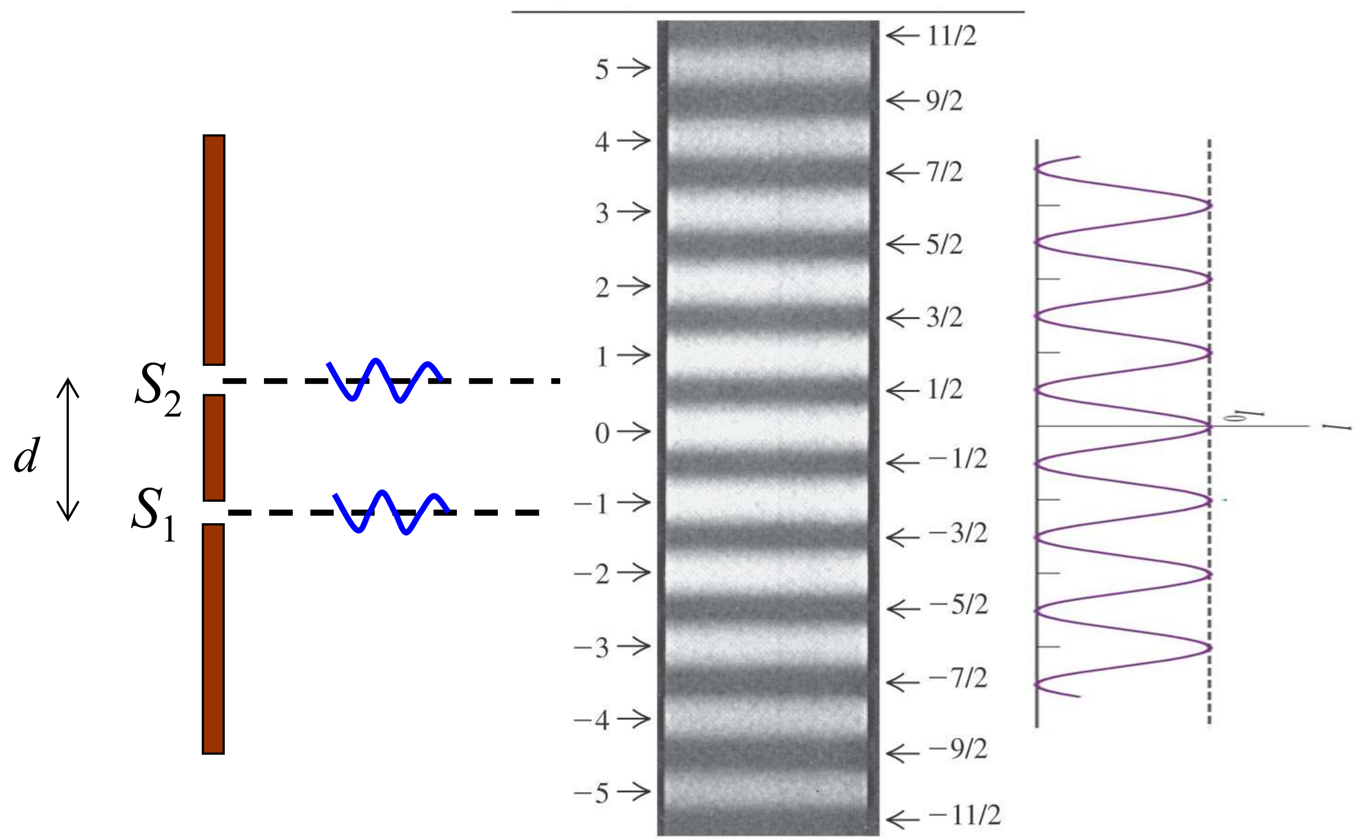
Constructive Interference:

$$d\sin\theta = m\lambda \quad (m = 0, \pm 1, \pm 2, \cdots)$$

Destructive Interference:

$$d\sin\theta = \left(m + \frac{1}{2}\right)\lambda \quad (m = 0, \pm 1, \pm 2, \cdots)$$

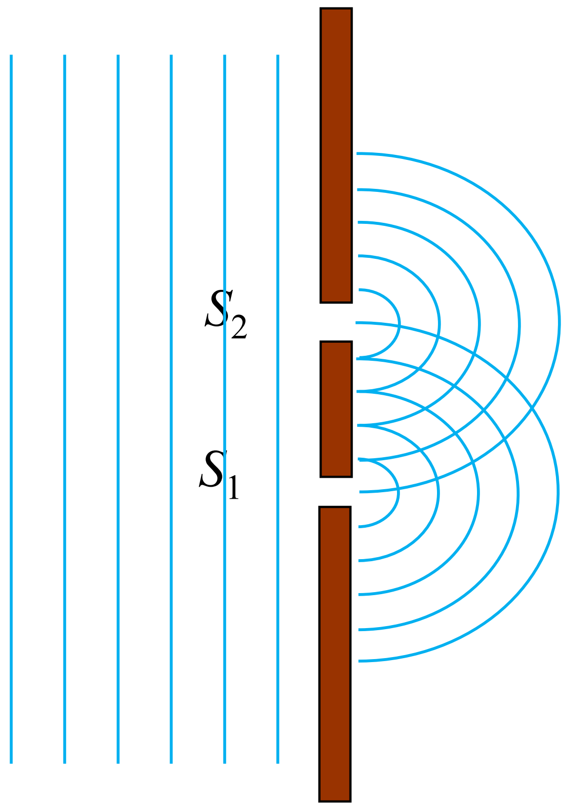
Double-Slit Interference Pattern (w/o diffraction)



$$I = I_0 \cos^2\left(\frac{\pi}{\lambda} d \sin\theta\right)$$

Intensity of Two-Slits Diffraction Patterns

With two slits, we have *diffraction* from the individual slits **and** *interference* from the two slits.



Intensity of Two-Slits Diffraction Patterns

With two slits, we have *diffraction* from the individual slits **and** *interference* from the two slits.

The *combined* intensity is the *superposition* of the two effects:

$$I = I_0 \cos^2\left(\frac{\phi}{2}\right)\left[\frac{\sin(\beta/2)}{\beta/2}\right]^2$$

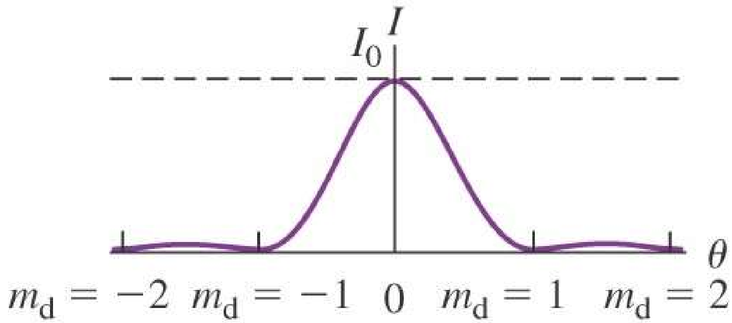
$$\underbrace{\cos^2\left(\frac{\phi}{2}\right)}_{\text{interference factor}} \qquad \underbrace{\left[\frac{\sin(\beta/2)}{\beta/2}\right]^2}_{\text{diffraction factor}}$$

where, $\frac{\phi}{2} = \frac{\pi}{\lambda} d \sin\theta$ and $\frac{\beta}{2} = \frac{\pi}{\lambda} a \sin\theta$

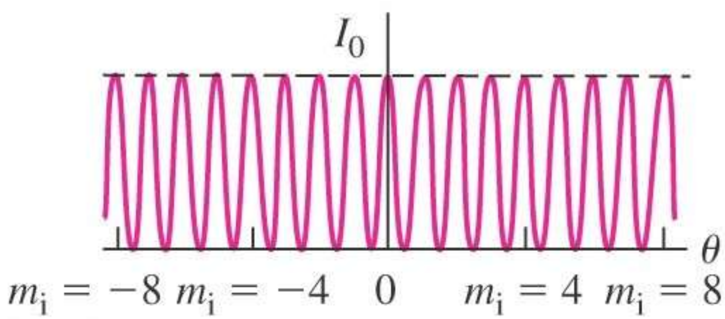
$d \rightarrow$ separation bet. slits

$a \rightarrow$ width of both slits

Intensity of Two-Slits Diffraction Patterns



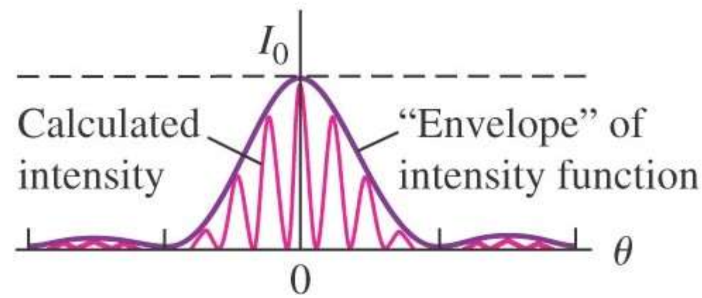
$\otimes$

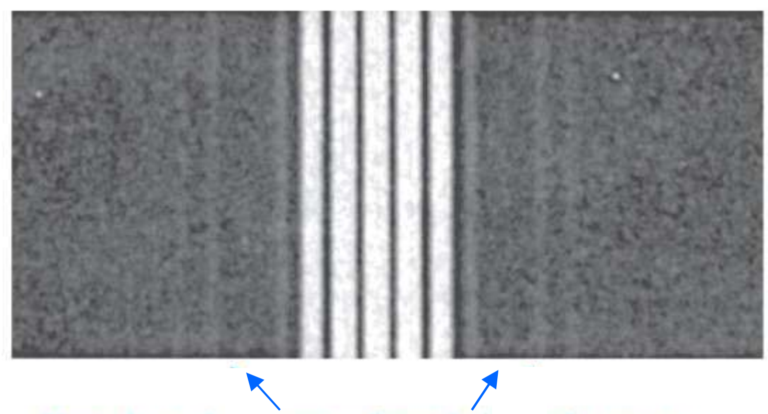


$$d = 4a$$

$=$

$$I = I_0 \cos^2\left(\frac{\phi}{2}\right)\left[\frac{\sin(\beta/2)}{\beta/2}\right]^2$$

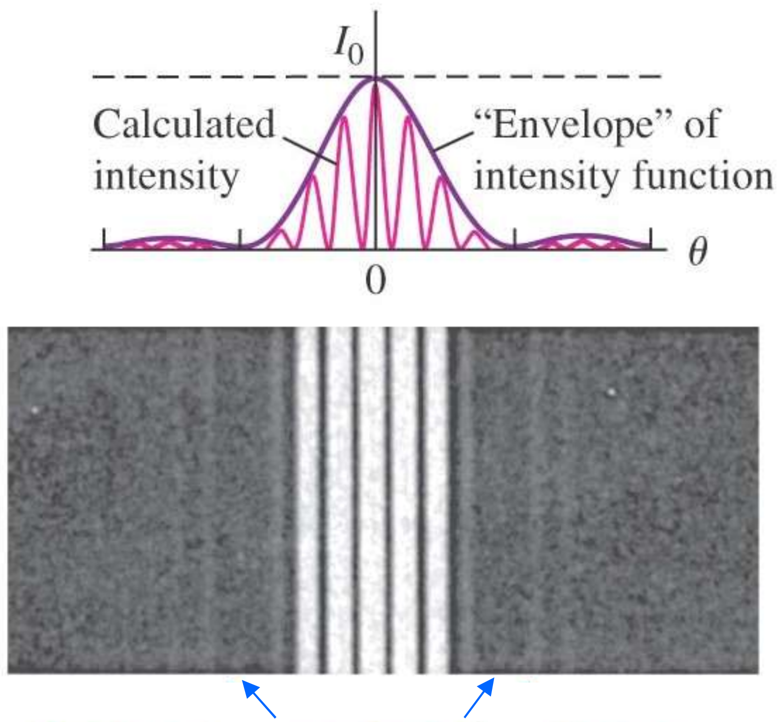




For $d = 4a$, every fourth interference maximum at the sides ($m_i = \pm 4, \pm 8, \ldots$) is missing,

Dmathematica D double slit

Intensity of Two-Slits Diffraction Patterns



For $d = 4a$, every fourth interference maximum at the sides ($m_i = \pm 4, \pm 8, \ldots$) is missing,

$$d = 4a$$

$$I = I_0 \cos^2\left(\frac{\phi}{2}\right)\left[\frac{\sin(\beta/2)}{\beta/2}\right]^2$$

Interfer max: $d \sin\theta = m\lambda$

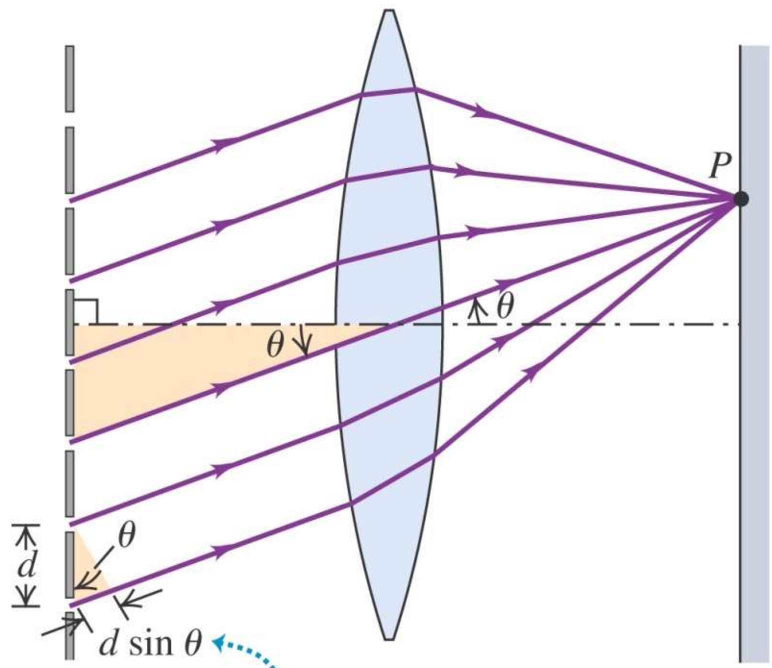
diff min: $a \sin\theta' = m'\lambda$

When do they match?

$$\sin\theta = \sin\theta' \Rightarrow \frac{m}{m'} = \frac{d}{a} = 4$$

Interference Patterns from Multiple Slits

Let consider an example with
N=8 slits,



Maxima occur where the path difference for adjacent slits is a whole number of wavelengths: $d \sin\theta = m\lambda$.

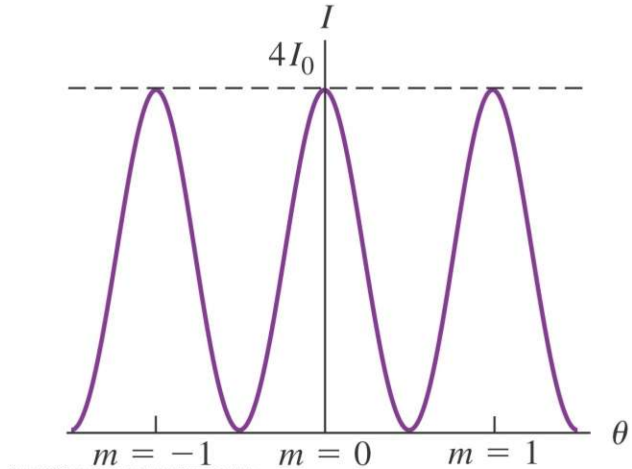
On the screen at P, **maximum** will occur at:

$$d\sin\theta = m\lambda, \quad m = 0, \pm 1, \pm 2, \cdots$$

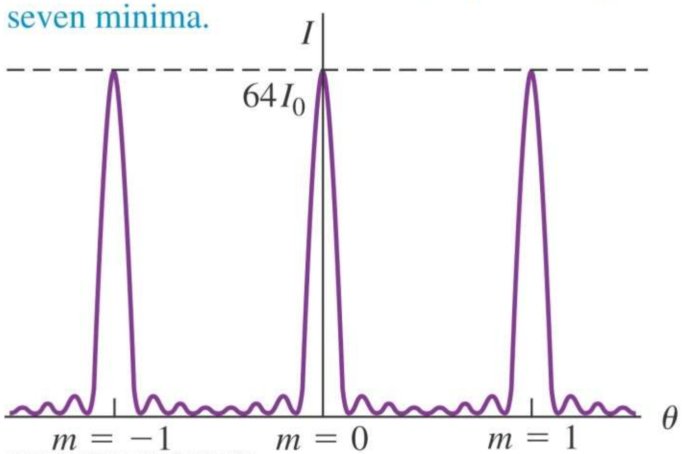
- when waves from adjacent slits have a path difference which is exactly $m\lambda$.
- This condition for maximums is the same for the two-slits patterns.

Interference Patterns from Multiple Slits

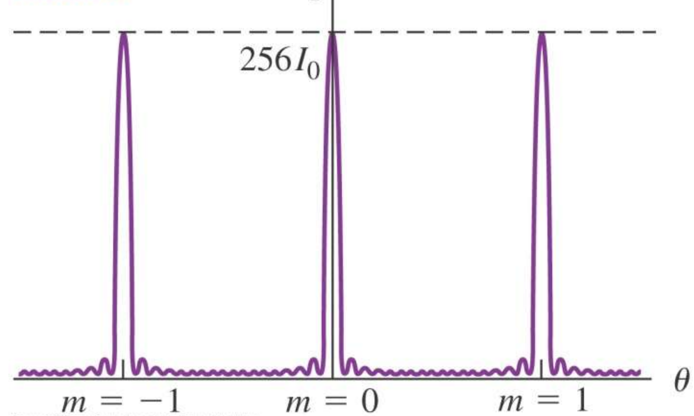
(a) $N = 2$: two slits produce one minimum between adjacent maxima.



(b) $N = 8$: eight slits produce taller, narrower maxima in the same locations, separated by seven minima.



(c) $N = 16$: with 16 slits, the maxima are even taller and narrower, with more intervening minima.



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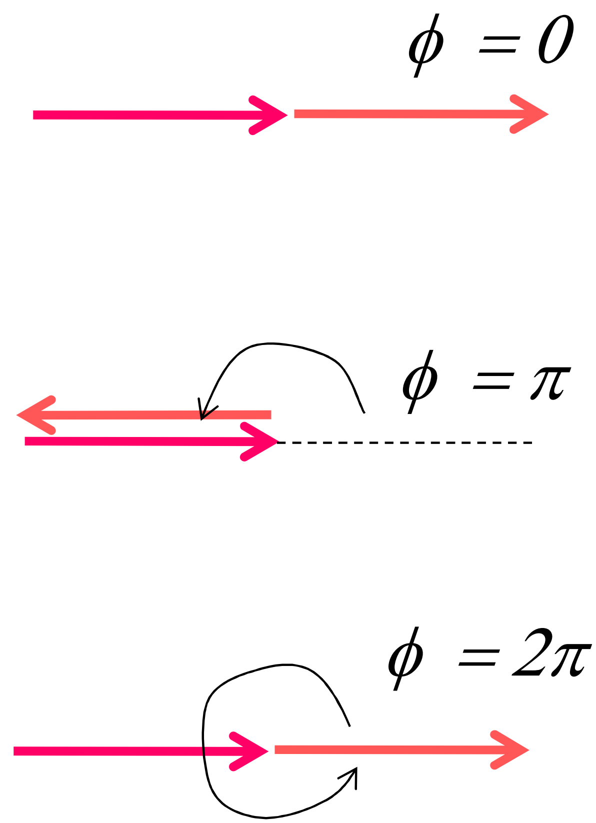
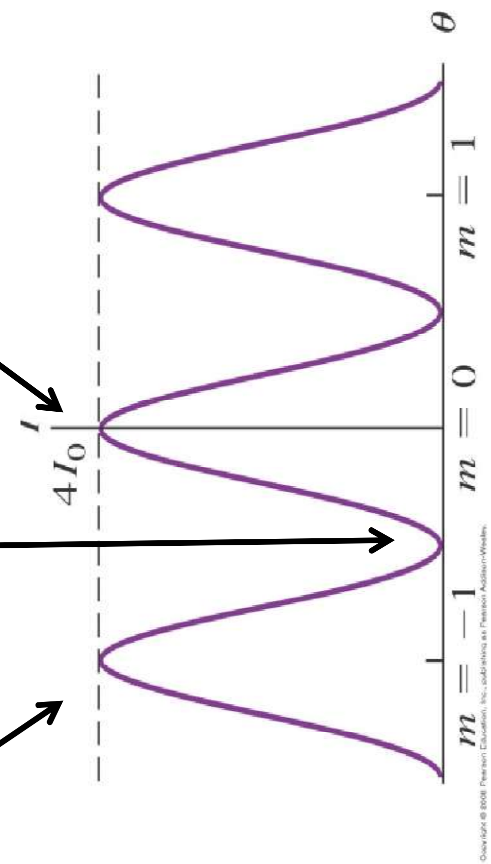
Interference Patterns from Two Slits



Central max

1st order min

1st order max

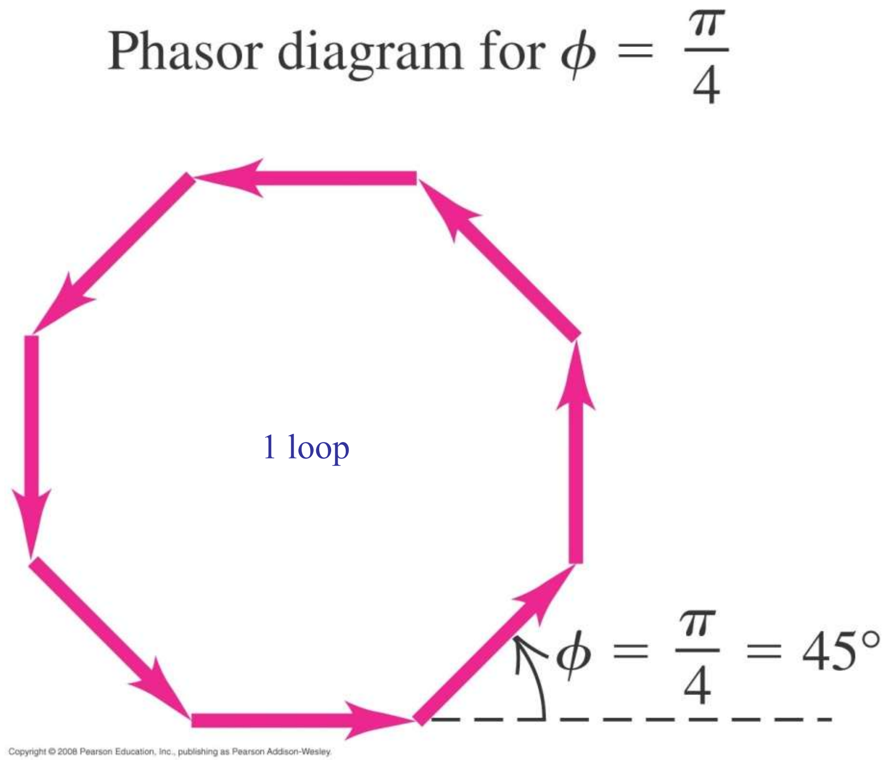


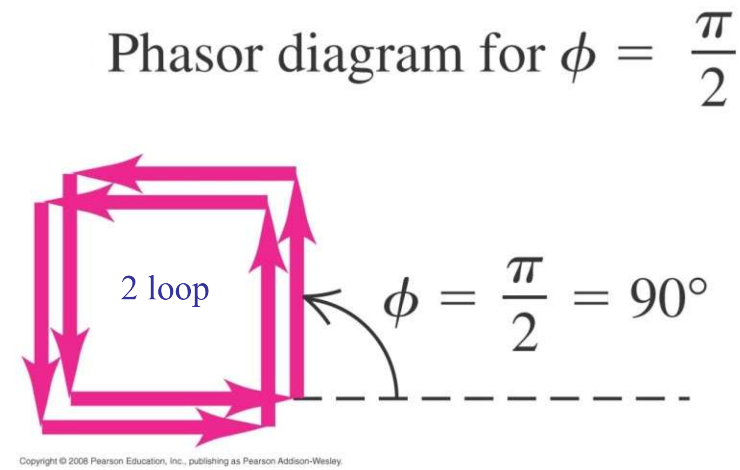
For $N = 2$, there is a single minimum exactly ½ way between 0 and 2π.

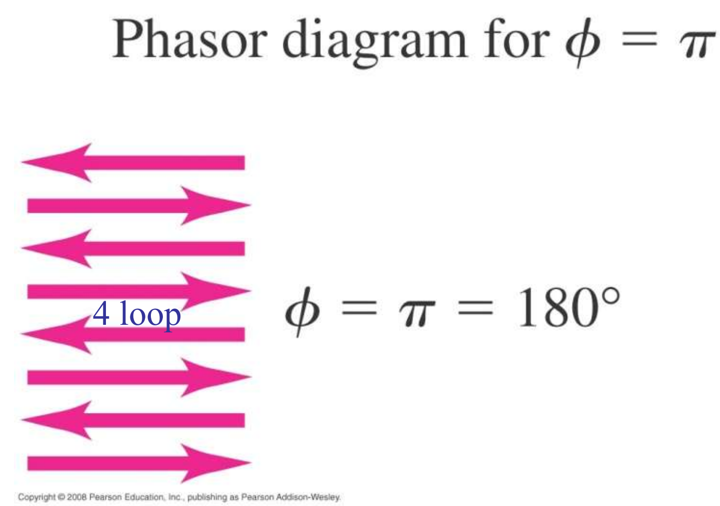
at $\phi = \pi = \frac{2\pi}{2} = \frac{2\pi}{N}$

Interference Patterns from Several Slits

Now, let look at the condition for the **minima** when ϕ goes from 0 to 2π for N=8:







For $N = 8$, there are a total of 7 minima with

$$\phi = \frac{\pi}{4}, \frac{2\pi}{4}, \frac{3\pi}{4}, \frac{4\pi}{4}, \frac{5\pi}{4}, \frac{6\pi}{4}, \frac{7\pi}{4} = m\frac{2\pi}{N}, m = 1, \ldots, N-1$$