Intensity of Interference Patterns

Wish to find $I(\theta)$ on a screen far away …

Let consider the $E$ fields coming from the double slits:

$E$ field from $S_2$ has a phase lag $\phi$ due to the extra path difference, $r_2 - r_1$.

$E_2(t) = E \cos(\omega t + \phi)$

$E_1(t) = E \cos(\omega t)$
Intensity of Interference Patterns

**Phasor Representation of an $E$ Field:**

- $E$ field as a vector (phasor) $\vec{E}$ rotating in the x-y plane with an angular frequency $\omega$.
- The time variation of this $E$ field, $E(t)$, is given as the horizontal projection (light red) of the phasor $\vec{E}$ (dark red).

\[ E(t) = E_0 \cos(\omega t) \]
Phasor in Action

- \( E(t) = E_0 \cos(\omega t) \)

- \( E \) field as a vector (phasor) rotating in the x-y plane with an angular frequency \( \omega \).

- The time variation of this \( E \) field, \( E(t) \) is given as the horizontal projection (light red) of the phasor (dark red).

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Recall that there are two coherent $E$ fields with a slight *phase difference* coming from the double slits:

\[ E_1(t) = E \cos(\omega t) \]

\[ E_2(t) = E \cos(\omega t + \phi) \]

$E$ field from $S_2$ has a **phase lag** $\phi$ due to the extra **path difference**, $r_2 - r_1$. 

Intensity of Interference Patterns
Phase Difference relates to Path Difference

Here, we have the lighter cyan wave slightly ahead of the blue wave.

\[ \frac{\phi}{2\pi} = \frac{r_2 - r_1}{\lambda} \]

This gives the relation,

\[ \phi = \frac{2\pi}{\lambda} (r_2 - r_1) = k (r_2 - r_1) \]

where \( k = \frac{2\pi}{\lambda} \) is called the wave number.
Phase Difference depends on Path Difference

Recall from our geometry, we have the following picture for the path difference:

\[ r_2 - r_1 = d \sin \theta \]

Substituting this into our previous equation, we have:

\[ \phi = \frac{2\pi}{\lambda} (r_2 - r_1) = \frac{2\pi d}{\lambda} \sin \theta \]

NOTE: We expressed one full cycle as \(2\pi\) so that \(\phi\) has to be in radian!
Intensity of Interference Patterns

At a given point on the screen far away from the two slits, the total $E$ field at $P$, $E_P$, is given by the vector-sum of the two phasors $\vec{E}_1$ and $\vec{E}_2$.

To find the magnitude of the resultant phasor $\vec{E}_P$, $E_P$, we use the law of cosines.

$$E_P^2 = E^2 + E^2 - 2E^2 \cos(\pi - \phi)$$
Intensity of Interference Patterns

Using the symmetry of the cosine function, \( \cos(\pi - \phi) = -\cos \phi \)

we have, \( E_P^2 = E^2 + E^2 - 2E^2 \cos(\pi - \phi) \)

\[
E_P^2 = 2E^2 + 2E^2 \cos \phi
\]

\[
E_P^2 = 2E^2 (1 + \cos \phi)
\]

Using another trig identity, \( 1 + \cos \phi = 2 \cos^2 \left( \frac{\phi}{2} \right) \)

we have, \( E_P^2 = 4E^2 \cos^2 \left( \frac{\phi}{2} \right) \). This gives, \( E_P = 2E \left| \cos \left( \frac{\phi}{2} \right) \right| \).
Intensity of Interference Patterns

The intensity of an electromagnetic wave is given by the average magnitude of the Poynting vector, $S_{av}$.

In general, the Poynting vector is proportional to the square of the magnitude of the electric field so for the intensity at $P$,

We can write the expression as,

$$ I = I_0 \cos^2 \left( \frac{\phi}{2} \right) $$

where $I_0$ is the maximum intensity when $\phi = 0$.

Note: when the two waves are in phase ($\phi = 0$, straight ahead), the resultant intensity is at maximum ($I=I_0$) and when the two waves are exactly half-cycle out of phase ($\phi = \pi$), the resultant intensity is identically zero.
Intensity in Two-Slit Interference

Putting this expression for the phase difference into our previous intensity equation for a two-slit interference pattern, we then have,

\[ I = I_0 \cos^2 \left( \frac{\phi}{2} \right) = I_0 \cos^2 \left( \frac{1}{2} \frac{2\pi d}{\lambda} \sin \theta \right) \]

\[ I = I_0 \cos^2 \left( \frac{\pi d}{\lambda} \sin \theta \right) \]
From the intensity equation, we can re-derive the conditions for the bright (maximum) and dark (minimum) fringes:

**Maximum** occurs when: \( \frac{\pi d}{\lambda} \sin \theta = m\pi \quad \rightarrow \quad d \sin \theta = m\lambda \quad (m = 0, \pm 1, \cdots) \)

**Minimum** occurs when: \( \frac{\pi d}{\lambda} \sin \theta = (m + \frac{1}{2})\pi \quad \rightarrow \quad d \sin \theta = (m + \frac{1}{2})\lambda \quad (m = 0, \pm 1, \cdots) \)
Interference in Thin Films

Color fringes observed from an oil slick on water or on a soap bubble are the white-light *interference* patterns produced by the reflected light off a *thin film* of oil or soap.

Light reflected from the upper and lower surfaces of the film comes together in the eye at point *P* and undergoes interference.

Some colors interfere constructively and others destructively, creating the color bands we see.
Phase Shifts During Reflection

From Maxwell’s Equations, one can show that the reflected wave will suffer a $180^\circ$ or $\lambda/2$ phase shift if it is reflected off from a medium with a larger $n$.

$$E_r = \frac{n_a - n_b}{n_a + n_b} E_i$$  \hspace{0.5cm} \text{(for normal incidence)}
Interference from a Thin Air Gap

**Assumptions:**
- Thickness of air gap $t$ is small
- Thickness of glass is large
- Incident light is nearly *normal* at the upper plate.

wave #1: reflected from top interface of air gap:

$$n_{\text{air}} < n_{\text{glass}} \Rightarrow \text{no phase shift}$$

wave #2: reflected from bottom interface of air gap:

$$n_{\text{glass}} > n_{\text{air}} \Rightarrow 180^\circ \text{ (or } \pi \text{) phase shift}$$
Interference from a Thin Air Gap

Now, consider the conditions for interference:

**Constructive**: wave #1 and wave #2 upon reflection must have a net phase difference of multiples of $2\pi$, i.e.,

$$m(2\pi), \quad m = 0, 1, 2, \ldots$$

- wave #1: suffers no phase shift during reflection
- wave #2: acquires a $\pi$ ($180^\circ$) phase shift during reflection and it also gains an extra phase shift due to path difference $= 2t$ in the air gap.

So, the net phase diff accumulated between wave #1 and #2:

$$\frac{2t}{\lambda} (2\pi) + \pi = m(2\pi), \quad m = 1, 2, 3, \ldots$$

For const interf,

$$\frac{2t}{\lambda} (2\pi) = \pi, 3\pi, \ldots$$
Interference from a Thin Air Gap

Solving for only positive $t$’s, we have \[ 2t = \frac{\lambda}{2}, \frac{3\lambda}{2}, \frac{5\lambda}{2}, \ldots \]

Rewriting, \[ 2t = \left( m + \frac{1}{2} \right)\lambda, \quad m = 0, 1, 2, \ldots \]

(condition for Con. Int. from thin film where one of the waves suffers a phase shift)

**Destructive**: wave #1 and wave #2 upon reflection must have a *net* phase difference of \[ (m+1/2)(2\pi), \quad m = 0, 1, 2, \ldots \]

Again, the net phase diff accumulated between wave #1 and #2 is \[ \frac{2t}{\lambda}(2\pi) + \pi \]

\[ \frac{2t}{\lambda}(2\pi) + \pi = \left( m + \frac{1}{2} \right)(2\pi), \quad m = 0, 1, 2, \ldots \]

\[ = \pi, 3\pi, 5\pi, \ldots \]
Interference from a Thin Air Gap

Rearranging the equation, we have,
\[
\frac{2t}{\lambda} (2\pi) + \pi = \pi, 3\pi, 5\pi \ldots
\]
\[
\frac{2t}{\lambda} (2\pi) = 0, 2\pi, 4\pi
\]

Finally, we have \(2t = m\lambda, \ m = 0, 1, 2, \ldots\) (condition for Des. Int. from thin film where one of the waves suffers a phase shift)
Thin and Thick Films

(a) Light reflecting from a thin film

Bursts of light a few μm long

The waves reflected from the two surfaces are part of the same burst and are coherent.

(b) Light reflecting from a thick film

The waves reflected from the two surfaces are from different bursts and are not coherent.

Interference effects can be observed

Interference effects are difficult to observe
Another Thin Film Example (non-refractive coating on lens)

wave #1: reflected from top interface of the coating:

\[ n_{\text{coating}} > n_{\text{air}} \implies 180^\circ \text{ (or } \pi \text{) phase shift} \]

wave #2: reflected from bottom interface of the coating:

\[ n_{\text{glass}} > n_{\text{coating}} \implies 180^\circ \text{ (or } \pi \text{) phase shift} \]
Interference from a Thin Film

Since both wave #1 and #2 suffers the same phase shift upon reflection, the net phase difference will be from the path difference \((2t)\) only. So, we have the standard condition (net phase diff. due to path diff. only),

There is one more consideration: the path difference is accumulated in a medium with \(n_{coating}\) so that the relevant wavelength should be \(\lambda_n = \lambda/n_{coating}\).

**Constructive:**
\[
2t = m\lambda_n, \quad m = 0,1,2,\ldots
\]
\[
2n_{coating} t = m\lambda, \quad m = 0,1,2,\ldots
\]

**Destructive:**
\[
2t = \left(m + \frac{1}{2}\right)\lambda_n, \quad m = 0,1,2,\ldots
\]
\[
2n_{coating} t = \left(m + \frac{1}{2}\right)\lambda, \quad m = 0,1,2,\ldots
\]

Note: In addition to wavelength modification, the RHS dependence are switched with respect to the air gap case.
Example 35.4: Thin Film

Question:
  a. Will there be a bright or dark fringe at the point of contact?
  b. What is the distance $x$ to the next bright fringe?
Example 35.4

Since $n_{\text{plastic}} < n_{\text{silicone}}$ and $n_{\text{silicone}} < n_{\text{glass}}$

*Both* wave #1 and #2 suffer a phase shift.

So, at the point of contact ($t=0$), the reflected wave #1 and #2 will arrive at the eyes *in phase* (bright fringe).

To find the location of the next constructive interference, we use,

\[ 2n_{\text{silicone}} t = \lambda_0 \ (m = 1) \]
Example 35.4

From the two similar triangles, we have

\[
\frac{h}{l} = \frac{t}{x} \quad \rightarrow \quad t = \frac{hx}{l}
\]

Substitute \( t \) into the previous eq,

\[
2n_{\text{silicone}}t = 2n_{\text{silicone}} \frac{hx}{l} = \lambda_0
\]

\[
x = \frac{l\lambda_0}{2n_{\text{silicone}}h} = \frac{(100\,mm)(500 \times 10^{-6}\,mm)}{2(1.50)(0.0200\,mm)} = 0.833\,mm
\]
Newton’s Rings

When viewed in monochromic light, the interference pattern is a set of concentric rings called the Newton’s rings.

Since each fringe corresponds to a path difference $\sim \lambda$, the lack of symmetry of these rings can be used to check for precision in lens making extremely accurately.
Distances comparable to \( \lambda \) can be measured with ease using this device by counting fringes.