Thin Lenses

Consider a thin lens as two closely spaced spherical surfaces.

• “thin” means that $t \ll$ other lengths
• For images produced by these two refracting surfaces, we will use the image $Q'$ from the first surface as the object for the second surface
Thin Lenses

Consider a *thin* lens as two closely spaced spherical surfaces.
Thin Lenses

LEFT refracting surface of thin lenses

\[ \frac{n_a}{s_1} + \frac{n_b}{s_1'} = \frac{n_b - n_a}{R_1} \]
Thin Lenses

RIGHT refracting surface of thin lenses

\[ \frac{n_b}{s_2} + \frac{n_c}{s_2'} = \frac{n_c - n_b}{R_2} \]

Note: \(Q'\) is now a virtual object for the right refracting surface.

From how we set up the problem, LEFT is still the incoming side.
For the situation indicated here, $Q'$ is on the side of the "out-going" light. By the sign convention, we have: $s_1' (image \ dist \ 1) > 0$ and $s_2 (object \ dist \ 2) < 0$

But, since they represent the same physical distance to $Q'$, for consistency, we need to have:

$$s_2 = -s_1'$$
Thin Lens

\[
\frac{n_a}{s_1} + \frac{n_b}{s_1'} = \frac{n_b - n_a}{R_1}
\]

(left surface)

\[
\frac{n_b}{s_2} + \frac{n_c}{s_2'} = \frac{n_c - n_b}{R_2}
\]

(right surface)
Thin Lens

\[ \frac{n_a}{s_1} + \frac{n_b}{s_1'} = \frac{n_b - n_a}{R_1} \] (left surface)

\[ \frac{n_b}{s_2} + \frac{n_c}{s_2'} = \frac{n_c - n_b}{R_2} \] (right surface)

NOTES:
- Since both \( C_1 \) and \( C_2 \) are on the outgoing side of light, \( R_1 \) and \( R_2 \) are + by convention.
- Since the material outside of the lens is typically air or vacuum, we take \( n_a \) and \( n_c = 1 \).
- For simplicity, we will call \( n_b \) (for the lens itself) \( n \).
- We also apply the image-to-object consistency relation: \( s_2 = -s_1' \)

\[ \frac{1}{s_1} + \frac{n}{s_1'} = \frac{n - 1}{R_1} \]

\[ -\frac{n}{s_1'} + \frac{1}{s_2'} = \frac{1 - n}{R_2} \]
Thin Lens

To eliminate $s_1'$ by adding these two equations:

$$\frac{1}{s_1} + \frac{1}{s_2'} = (n-1) \left( \frac{1}{R_1} - \frac{1}{R_2} \right)$$

Calling our original object distance $s_1$ simply as $s$ and our final image distance $s_2'$ simply as $s'$, we have:

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f} \quad \text{(object-image relation, thin lens)}$$

where,

$$\frac{1}{f} = (n-1) \left( \frac{1}{R_1} - \frac{1}{R_2} \right) \quad \text{(lensmaker’s equation)}$$
Converging & Diverging lens

Depending on the values of \( n_{\text{lens}} \), \( n_{\text{outside}} \), \( R_1 \) and \( R_2 \), \( f \) can be positive or negative!

\[
\frac{1}{f} = \left( \frac{n_{\text{lens}} - n_{\text{outside}}}{n_{\text{outside}}} \right) \left( \frac{1}{R_1} - \frac{1}{R_2} \right)
\]

- \( f \rightarrow \text{positive} \) \hspace{1cm} Converging lenses
- \( f \rightarrow \text{negative} \) \hspace{1cm} Diverging lenses

Meniscus  \hspace{1cm} Planoconvex  \hspace{1cm} Double convex  \hspace{1cm} Meniscus  \hspace{1cm} Planoconcave  \hspace{1cm} Double concave
Sign Rules for Mirrors & Lens

1. **Object Distance:**
   - $s$ is $+$ if the object is on the same side as the incoming light (for both reflecting and refracting surfaces) and $s$ is $-$ otherwise.

2. **Image Distance:**
   - $s'$ is $+$ if the image is on the same side as the outgoing light and is $-$ otherwise.

3. **Object/Image Height:**
   - $y$ ($y'$) is $+$ if the image (object) is erect or upright. It is $-$ if it is inverted.

4. **Radius of Curvature:**
   - $R$ is $+$ when the center of curvature $C$ is on the same side as the outgoing light and $-$ otherwise.

5. **Focus Length:**
   - (+ concave, - convex)
   - (+ converging, - diverging)
Consider a far away object \((s = \infty)\) so that the incoming rays are parallel to the optical axis, these rays will all converge at one point (the right focal point \(F_2\)) at a distance \(f\) (the focal length) to the right of the lens,

\[ f > 0 \quad (converging) \]

\[
\frac{1}{\infty} + \frac{1}{s'} = \frac{1}{f} \quad \rightarrow \quad s' = f
\]

This gives the focusing capability of a thin lens.
Focal Points of a Thin Lens

Now, consider the reverse situation, if the outgoing rays are all parallel to the optical axis so that the image is at infinity \((s' = \infty)\), where is the object originally located?

\[
\frac{1}{s} + \frac{1}{\infty} = \frac{1}{f} \quad \rightarrow \quad s = f
\]

This simple calculation indicates that all rays must originate from a single point (the left focal point \(F_1\)) at a distance \(f\) (the focal length) to the left of the lens,
Properties of a Diverging Lens

\[ f < 0 \quad (\text{diverging}) \]

Second focal point: The point from which parallel incident rays appear to diverge

First focal point: Rays converging on this point emerge from the lens parallel to the axis.

For a diverging thin lens, \( f \) is negative.
Symmetry of the Lensmaker’s Equation

Because the Lensmaker’s equation is symmetric with respect to the sign convention for $R_a$ and $R_b$, the left and right focal lengths are the same irrespective of the difference in the values of $R_a$ and $R_b$.

Example: Application of the Lensmaker’s Equation:

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_a} - \frac{1}{R_b}\right)$$

$$= (1.33-1)\left(\frac{1}{10} - \frac{1}{15}\right) = \frac{0.33}{30} = 0.011$$

$$f = +90.9\text{cm}$$

$R_a = +10.0\text{cm}$ ($C_a$ same side as outgoing light) \[\text{The lens is converging.}\]

$R_b = +15.0\text{cm}$ ($C_a$ same side as outgoing light)
Symmetry of the Lensmaker’s Equation

Now, we flip our lens around so that the refracting surface with $R_b$ will be on the incoming light side,

\[
\frac{1}{f} = (n-1) \left( \frac{1}{R_b} - \frac{1}{R_a} \right)
\]

\[
= (1.33 - 1) \left( \frac{-1}{15} - \left( \frac{-1}{10} \right) \right)
\]

\[
= 0.33 \left( \frac{-1}{15} + \frac{1}{10} \right) = \frac{0.33}{30} = 0.011
\]

\[f = +90.9 cm\]

$R_a = -10.0 cm \ (C_a \ \text{NOT on outgoing light side})$

$R_b = -15.0 cm \ (C_a \ \text{NOT on outgoing light side})$

Same result as previous calculation!
\[ \Delta OPQ \text{ and } \Delta OP'Q' \text{ are similar, so that we have, } \frac{y}{s} = \frac{-y'}{s'} \]

This gives, \[ m = \frac{y'}{y} = \frac{-s'}{s} \] (lateral magnification, thin lens)
Ray Tracing Methods for Lenses

(a) Converging lens

1. Parallel incident ray refracts to pass through second focal point $F_2$.
2. Ray through center of lens does not deviate appreciably.
3. Ray through the first focal point $F_1$ emerges parallel to the axis.
Rays Tracing Methods for Lenses

(b) Diverging lens

Focal Point on the incoming side

1. Parallel incident ray appears after refraction to have come from the second focal point $F_2$.
2. Ray through center of lens does not deviate appreciably.
3. Ray aimed at the first focal point $F_1$ emerges parallel to the axis.
Object-Image Relations: Thin Lenses

**Thin Lens Applet (by Fu-Kwun Hwang)**
http://www.physics.metu.edu.tr/~bucurgat/ntnujava/Lens/lens_e.html

(a) Object $O$ is outside focal point; image $I$ is real.

(b) Object $O$ is closer to focal point; image $I$ is real and farther away.
Object-Image Relations: Thin Lenses

(c) Object $O$ is even closer to focal point; image $I$ is real and even farther away.

(d) Object $O$ is at focal point; image $I$ is at infinity.
Object-Image Relations: Thin Lenses

(e) Object $O$ is inside focal point; image $I$ is virtual and larger than object.

(f) A virtual object $O$ (light rays are converging on lens)
Example: Compound Lenses
Example: Component Lens

For the 1st Lens: \[ \frac{1}{s_1} + \frac{1}{s_1'} = \frac{1}{f_1} \rightarrow \frac{1}{12} + \frac{1}{s_1'} = \frac{1}{8} \rightarrow \frac{1}{s_1'} = \frac{12-8}{96} = \frac{1}{24} \]

\[ s_1' = 24\text{cm} \]

This image from the 1st lens is on the light incoming side of lens #2, so that:

\[ s_2 = 36\text{cm} - s_1' = +12\text{cm} \]

\[ \frac{1}{s_2} + \frac{1}{s_2'} = \frac{1}{f_2} \rightarrow \frac{1}{12} + \frac{1}{s_2'} = \frac{1}{6} \rightarrow \frac{1}{s_2'} = \frac{2-1}{12} = \frac{1}{12} \]

\[ s_2' = 12\text{cm} \]

(final image is 12 cm on the outgoing side of lens #2)

The combined lateral magnification is the product from both lenses,

\[ m_{tot} = m_1m_2 = \left(-\frac{s_1'}{s_1}\right)\left(-\frac{s_2'}{s_2}\right) = \frac{24\text{cm}}{12\text{cm}} \frac{12\text{cm}}{12\text{cm}} = +2 \] (upright and real)
Another Example of Component Lens

Two converging lens with \( f_1 = 20\text{cm} \) and \( f_2 = 10\text{cm} \) and the lens are 20 cm apart. Object is located 30cm to the left of lens #1, find the location of final image.
Another Example of Component Lens

For the 1\textsuperscript{st} Lens:

\[
\frac{1}{s_1} + \frac{1}{s_1'} = \frac{1}{f_1} \rightarrow \frac{1}{30} + \frac{1}{s_1'} = \frac{1}{20} \rightarrow \frac{1}{s_1'} = \frac{3-2}{60} = \frac{1}{60}
\]

\[s_1' = 60cm\]

This image from the 1\textsuperscript{st} lens is on the \textit{outgoing} side of lens #2, so that:

\[s_2 = -40cm\]

\[
\frac{1}{s_2} + \frac{1}{s_2'} = \frac{1}{f_2} \rightarrow -\frac{1}{40} + \frac{1}{s_2'} = \frac{1}{10} \rightarrow \frac{1}{s_2'} = \frac{4+1}{40} = \frac{1}{8}
\]

\[s_2' = 8cm\] (final image is 8 cm on the outgoing side of lens #2)

The \textit{combined} lateral magnification is the product from both lenses,

\[
m_{\text{tot}} = m_1m_2 = \left( -\frac{s_1'}{s_1} \right) \left( -\frac{s_2'}{s_2} \right) = -\frac{60cm}{30cm} \left( -\frac{8cm}{-40cm} \right) = -0.4 \quad \text{(inverted and real)}
\]
Fermat’s Principle
Pierre de Fermat (1601-1665)

A general mathematical principle that can be used to analyze light path:

“When a light ray travels between two points, its path is the one that requires the least time.”

Application #1: uniform material \([n \text{ (or } v) \text{ is the same everywhere!}]\)

\[ t = \frac{d}{v} \rightarrow \] Between any two points, the least time requires the shortest distance in an uniform medium.

Light will travel in a straight line in an uniform medium.
Application #2: Snell’s Law

Within $n_1$ and $n_2$, light travels in straight lines and total time of travel from $p$ to $q$ is,

$$t = \frac{r_1}{v_1} + \frac{r_2}{v_2} = \frac{\sqrt{a^2 + x^2}}{c/n_1} + \frac{\sqrt{b^2 + (d-x)^2}}{c/n_2}$$

Note: With two different speeds, the fastest way to get from $p$ to $q$ is not necessary a straight line!
Fermat’s Principle
(Application to Snell’s Law)

\[ t(x) = \frac{\sqrt{a^2 + x^2}}{c/n_1} + \frac{\sqrt{b^2 + (d-x)^2}}{c/n_2} \]

Find the value of \( x \) (the crossing point) such that the total travel time is minimized.

\[ \frac{dt}{dx} = 0 \rightarrow \frac{n_1}{c} \left( \frac{1}{2} \right) \frac{2x}{\sqrt{a^2 + x^2}} + \frac{n_2}{c} \left( \frac{1}{2} \right) \frac{2(d-x)(-1)}{\sqrt{b^2 + (d-x)^2}} = 0 \]
Fermat’s Principle
(Application to Snell’s Law)

\[
\frac{n_1 x}{\sqrt{a^2 + x^2}} - \frac{n_2 (d-x)}{\sqrt{b^2 + (d-x)^2}} = 0
\]

\[r_1^2 = a^2 + x^2\]
\[r_2^2 = b^2 + (d-x)^2\]
\[
\sin \theta_1 = \frac{x}{r_1}
\]
\[
\sin \theta_2 = \frac{(d-x)}{r_2}
\]

\[n_1 \sin \theta_1 - n_2 \sin \theta_2 = 0\]

\[n_1 \sin \theta_1 = n_2 \sin \theta_2\quad \text{(Snell’s Law)}\]