Qualifying exam - January 2013

Statistical Mechanics

You can use one textbook. Please write legibly and show all steps of your derivations.

Problem 1 [20 points]
For a particular substance it was found that

$$\left(\frac{\partial V}{\partial T}\right)_p = Ape^{-ap}, \tag{1}$$

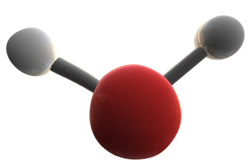
where V is volume, p is pressure, and the coefficients A and a depend only on temperature. Consider an equilibrium process in which the pressure of the substance increases from 0 to p_1 at a constant temperature T.

1. What is the change in entropy of the substance?
2. Compute the amount of heat received by the substance in this process.

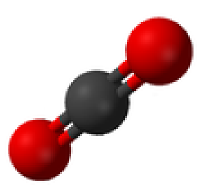
Problem 2 [20 points]
An adiabatic rigid cylinder is divided in two compartments by a piston. One compartment is filled with $N_1 = 13$ moles of water vapor at a temperature T_1 and pressure p_1. The other compartment is filled with $N_2 = 12$ moles of carbon dioxide CO_2 at a temperature $T_2 = 3T_1$ and pressure $p_2 = 2p_1$. Each gas is initially in thermodynamic equilibrium. The piston is removed and the gases mix. After equilibrium has been reached,

1. What is the internal energy of the gas mixture in the cylinder?
2. What is the temperature of the gas mixture?
3. What is the pressure of the gas mixture?

Consider both gases as ideal and treat the molecular rotations and atomic vibrations using classical mechanics.



H_2O



CO_2

Problem 3 [10 points]
Consider a system of $N \gg 1$ weakly interacting particles, each having two energy levels: zero with a degeneracy g_1 and ε with a degeneracy g_2. The particles are identical but distinguishable.

1. Calculate the specific heat C_v of this system.
2. Find the asymptotic expressions for C_v in the limits of high temperatures and low temperatures.
3. Sketch C_v as a function of temperature.

Problem 4 [25 points]
Consider a quantum gas of ultra-relativistic particles (bosons or fermions) with the energy-momentum relation $\varepsilon = cp$, where c is the speed of light. Show that regardless of temperature

$$PV = \frac{E}{3}, \tag{2}$$

where P is pressure, V is volume of the gas and E is its total energy.

Does this result remain valid for an ultra-relativistic gas in the Maxwell-Boltzmann statistics?

Problem 5 [25 points]
Calculate the average energy per phonon (total energy divided by the number of phonons) in a Debye solid in the limit of low temperatures, i.e. $T \ll T_D$ (T_D being the Debye temperature).

You may need to use Riemann's zeta function

$$\varsigma(n) = \frac{1}{(n-1)!}\int_0^\infty \frac{x^{n-1}}{e^x - 1}dx$$

with $\varsigma(2) = \pi^2/6$, $\varsigma(3) \approx 1.202$ and $\varsigma(4) = \pi^4/90$.