Quantum Mechanics Qualifying Exam, 2013

Note: This is an open book exam and you are allowed to bring Sakurai or Shankar's book on Quantum Mechanics. If a formula appears in the book, please use that as a starting point, there is no need to show the derivation of that formula.

(1) [10 points] (a) Show that for a one-dimensional real wave function $\psi(x)$, the expectation value of momentum $\langle p\rangle = 0$.
(b) Show that if $\psi(x)$ has mean momentum $p_0 = \langle p\rangle$, then $e^{-ibx/\hbar}\psi(x)$ has mean momentum $\langle p\rangle = p_0 - b$.

(2) [20 points] Consider a particle of unit mass at absolute zero temperature. What is its energy if the particle is
(a) free;
(b) confined to a cubic box of unit dimension;
(c) confined to a potential $V(x, y, z) = V_0(2x^2 + (3y - y_0)^2 + 4z^2)$, where V_0 is a constant;
(d) subject to a potential $V(r, \theta, \phi) = -e^2/r$.

(3) [15 points] Consider the hydrogen atom.
(a) Write down the most general wave function of the electron in the first excited state. It should contain the radial and angular parts in terms of known special functions.
(b) If the electron is in the ground state, calculate the value of r at which the probability density is maximized.
(c) Write down the most general wave function of the electron that is spherically symmetric.

(4) [15 points] Consider a unitary operator $U = e^{iaG}$, where G is a Hermitian operator and a is a constant. Identify the G and a in the following cases
(a) $U|x\rangle = |x + x_0\rangle$
(b) $U|p\rangle = |p + p_0\rangle$
(c) $U|\alpha(t)\rangle = |\alpha(t + t_0)\rangle$
(d) $U|\theta, \phi\rangle = |\theta, \phi + \phi_0\rangle$

(e) $U|S_x, +\rangle = |S_y, +\rangle$
where x_0, p_0, t_0, ϕ_0 are all constants.

(5) [20 points] Consider a free particle of mass m and energy E moving in three dimension. Write down the wave function (or functions) of the particle if it is
(a) in an eigenstate of momentum.
(b) in an eigenstate of angular momentum with quantum number $l = 0$.
(c) in an eigenstate of angular momentum with quantum number $l = 1$.
(d) in a state in which $\Delta x \Delta p$ has minimum uncertainty.
(e) an eigenfunction of the parity operator.
(f) For (b) and (c), how does the wave function behave at $r = 0$.

(6) [20 points] A particle of mass m is confined by one-dimensional potential $V(x) = (5x - 3)^2$ for $x > 3/5$ and $V(x) = \infty$ for $x \leq 3/5$. At $t = t_0$, the particle is equally likely to be in the ground and the first excited state. What is the wave function of the particle? What will the wave function be at a later time $t = T$?