Quantum Mechanics Qualifying Exam

Jan. 2011

You are allowed to quote results directly from the main text of Shankar, but not from its exercises or solutions.

1. A particle of mass m is confined within a cubic box of side length L. The wave-function vanishes beyond the box.

a). What is the parity (eigenvalue) of the ground state? Assume the origin is at the center of the box. [5 pts]

b). What is the energy of the first excited state? What is its degeneracy? [10 pts]

c). Assume the particle is in its ground state. Suddenly, the box expands to a cube with side $2L$. What is the probability of finding the particle in the new ground state? [10 pts]

d). Consider 3 non-interacting identical bosons in the box, what is the energy of the first excited state? [5 pts]

2. A molecule is rotating around its center of mass. The Hamiltonian is $H = (L_x^2 + L_y^2)/2I_a + L_z^2/2I_b$, where $I_{a,b}$ are the moments of inertia, and $L_{x,y,z}$ are the orbital angular momentum operators.

a). Find the energy eigenvalues and eigenstates. [10 pts]

Now consider a state described by angular wave function $\psi(\theta, \phi) = \sqrt{3/4\pi} \sin\theta \cos\phi$, where θ and ϕ are the polar and azimuthal angles respectively. Using the Dirac ket notation $|lm\rangle$ may prove convenient for subsequent problems 2b) and 2c).

b). Compute the expectation value of L_z in state ψ. [10 pts]

c). Suppose L_z is measured in state ψ and result $\hbar$ is obtained. Immediately afterwards, L_x is measured, find the uncertainty (standard deviation) ΔL_x. [10 pts]

3. In April (O'Connell et al, Nature 464, 697, 2010), a group of physicists at UCSB succeeded in quantum control of a macroscopic mechanical system. A mechanical resonator made of aluminum nitride and aluminum was cooled to 25mK. A superconducting qubit was coupled to the resonator to prepare and measure its quantum states.

Treat the mechanical resonator as a one-dimensional harmonic oscillator of frequency ω. Let $|\psi_0\rangle$ and $|\psi_1\rangle$ be the normalized energy eigenstate of the ground state and the first excited state respectively. At time $t = 0$, the system is prepared at state $A|\psi_0\rangle + B|\psi_1\rangle$. A and B are in general complex numbers. Then, the dynamics of the system is monitored in the experiment to determine the relaxation and coherence time.

a). Compute the average value of energy E and position x at $t = 0$. [10 pts]

b). Find the state vector at later time $t > 0$. [10 pts]

c). What is the oscillation frequency of $\langle x\rangle$ as a function of time? [10 pts]

4. A boy drops a marble of mass m from height H (above the ground) and tries to hit a marked point on the ground. Show that no matter how hard he tries, the marble is going to miss the point by a distance Δx on the order of $(\hbar^2 H/m^2 g)^{1/4}$, where g is the gravitational acceleration. [10 pts]

Hint: the initial (horizontal) position and velocity are constrained by Heisenberg's uncertainty relation. Treat the motion of the marble after release as classical. This problem is a mock version of ^{87}Rb atoms released from an optical trap.