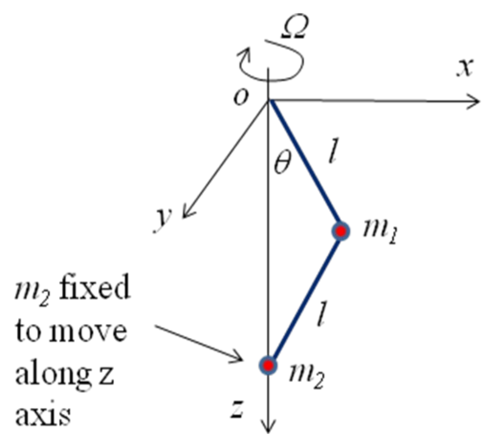
Classical Mechanics Qualifier (January 2011)
George Mason University

You will have **THREE** hours to complete the exam.
You MUST COMPLETE Problems 1 & 2.
CHOOSE TO COMPLETE one of the two among Problems 3 &4.
You are allowed to use your graduate textbook during the exam.

Problem 1 (30pts)
Two masses m_1, $m_2 = m$ are connected by two massless rigid rods of length l as shown. The top end of the rod is fixed at the origin O and m_2 is constrained to move along the z-axis. The whole system is rotating with a constant angular frequency Ω about the z-axis.



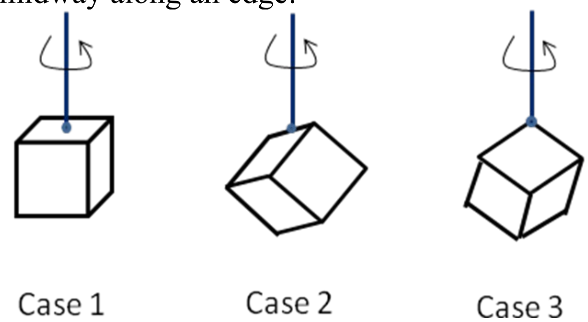
a) (10 pts) Choosing a convenient generalized coordinate, calculate the Lagrangian for the system. Use the Euler-Lagrange equation to derive an equation of motion for the system.
b) (10 pts) Find the three equilibrium states for this system. Explicitly specify the condition or conditions under which these equilibria can exist.
c) (10 pts) Examine the stability for these equilibria.

Problem 2 (30pts)
An orbit for an object in a central force problem is given by

$$r(\theta) = \frac{1}{a + b\cos(3\theta/2)}, \quad a,b > 0$$

a) (5 pts) Under what condition on a and b will the orbit be a bounded orbit?
b) (5 pts) What are the pericenter and apocenter for the bounded orbit?
c) (5 pts) What is the angular advance for the apsis from one apocenter to the next apocenter for this bounded orbit?
d) (5 pts) Sketched this bounded orbit. Is the orbit closed?
e) (5 pts) Determined the central potential $V(r)$ that produces this orbit.
f) (5 pts) What is the total energy E for the circular orbit in this system and what is the radius for this circular orbit?

Problem 3 (20pts)
A torsion pendulum consists of a vertical wire attached to a mass m which may rotate about the vertical axis. k is the torque constant for the wire. Consider three torsion pendula consisting of identical wires from which identical homogeneous solid cubes are hung. All cubes have side a. One cube is hung from the middle of a face, one from a corner, and one from midway along an edge:



a) (10 pts) Calculate the moment of inertia tensor for a cube with side a with respect to its center of mass.
b) (10 pts) What are the natural frequencies of oscillations for these three torsion pendulua. What is the relation between the natural frequencies for these three cases?

Problem 4 (20pts)

a) (5 pts)You are given the following transformation between two sets of phase space variables (x,p) and (Q,P)

$$Q = \frac{\alpha p}{x} \qquad P = \beta x^2, \qquad \text{where } \alpha, \beta \text{ are two real parameters}$$

What is the condition on α, β so that this transformation is canonical?

For the remainder of this problem, suppose $\beta = 1/2$.

b) (5 pts) You are given the Hamiltonian $K(Q,P)$,

$$K(Q,P) = \frac{PQ^2}{m} + kP$$

Find the Hamiltonian $H(x,p)$ under this canonical transformation. What physical system does this Hamiltonian represent?

c) (5 pts) Consider the following function:

$$u(x,p,t) = \ln(p + im\omega x) - i\omega t$$

Use the Poisson bracket to show that u is a constant of motion for a one-dimensional harmonic oscillator with natural frequency $\omega = \sqrt{k/m}$.

d) (5 pts) What does this constant of motion u correspond to physically?