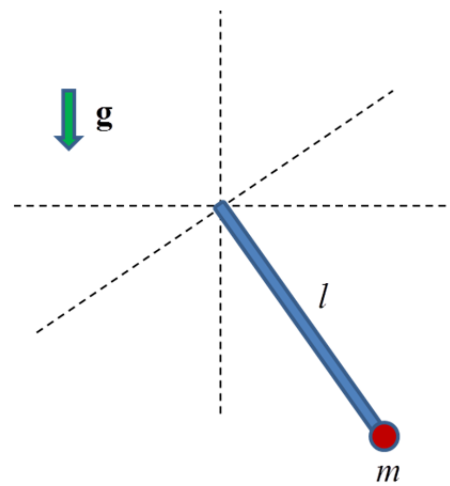
Classical Mechanics Qualifier (January 2014)
George Mason University

You will have **THREE** hours to complete the exam.
You are allowed to use your graduate textbook during the exam.
Choose **4 out of the 5** problems below.

Problem 1 (25pts)

A spherical pendulum made of a massless rod with length l and small mass m at the end is shown on the right.



a) Write down the Lagrangian for the system in spherical coordinates (r,θ,ϕ).

b) What are the cyclic variables and their associated conserved quanities for this pendulum?

c) Write down the equations of motion for the system.

d) Using the method of Lagrange multipliers, find an expression for the tension in the rod.

e) For a planar pendulum, what does the expression for the tension reduce to? In the limit of small angular displacements, what does the expression for the tension further reduce to?

Problem 2 (25pts)

A point particle moving around a black hole can be described by the following central force potential modified from the standard Keplerian case,

$$V_{BH}(r) = -\frac{1}{r} - \frac{l^2}{r^3}$$

where l is the angular momentum of the system. For simplicity, we have normalized the system so that $k = 1$ for the Keplerian term ($-k/r$) in the potential and $\mu = 1$ for the reduced mass.

a) Show that there are no circular orbits if $l^2 < 12$ and there are two if $l^2 > 12$.
b) Sketch a plot for the effective potential V_{eff} of the problem for the above two cases $l^2 < 12$ and $l^2 > 12$.
c) Describe qualitatively the set of possible orbits for the two different cases $l^2 < 12$ and $l^2 > 12$ with respect to the system's total energy *E*.

Problem 3 (25pts)

The Hamiltonian for a particular classical system is given by the following,

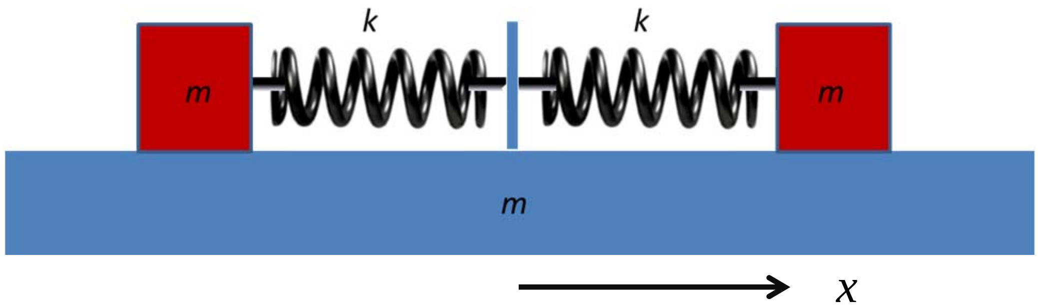
$$H = \frac{1}{2q^2} + \frac{p^2 q^4}{2}$$

a) Find the equations of motion for this system using the Hamilton's equations.
b) Find a generating function for the given canonical transformation:

$$Q(q,p) = -\frac{1}{q} \text{ and } P(q,p) = pq^2$$

c) Show that the transformed Hamiltonian $K(Q,P)$ corresponds to a harmonic oscillator?
d) With the initial conditions, $Q(0) = Q_0$ and $P(0) = 0$, find the solution to the equation of motion, $Q(t)$ and $P(t)$. Then, using the inverse canonical transformation, find $q(t)$ and $p(t)$.

Problem 4 (25pts)



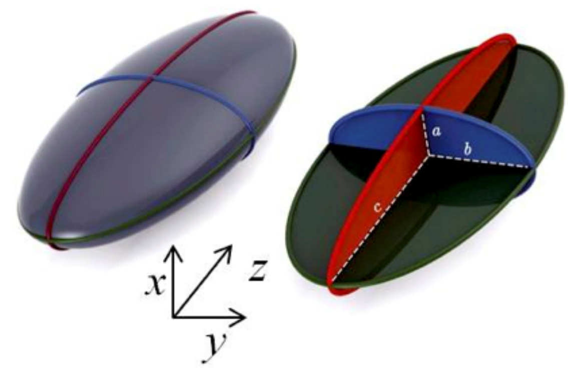
A platform of mass, *m*, is sitting on a frictionless surface and is free to move in the *x*-direction. Two identical blocks, also of mass, *m*, are connected to a thin massless post fixed to the platform by two identical massless springs of spring constant *k*. The blocks are free to move on the platform in the *x*-direction without friction.

a) Find the normal mode frequencies for the system in terms of *k* and *m*.

b) Describe the motion of each of the normal modes.

Problem 5 (25pts)

A spaceship lost its power in deep space (far away from any stars). The spaceship was originally spinning along one of its principal axes with a period T. A small perturbation to the angular velocity was seen to grow exponentially in time, $\delta\omega(t) \sim e^{\lambda t}$. Assume the spaceship to have a uniform density ρ, a total mass M, and an ellipsoidal shape with three semi-principal axes $a < b < c$ (see figure).



i) Calculate the Principal Moment of Inertia for the ellipsoid with respect to its center of mass. [Hint: If one rescales the axes, $x = au, y = bv, z = cw$, one can change the ellipsoid to a unit sphere for the integration, i.e.,

$$V = \int_{ellipsoid} dxdydz = \int_{sphere} (abc)dudvdw = \frac{4\pi}{3}abc \quad]$$

ii) Around which axis $(\hat{x}, \hat{y}, \hat{z})$ was the spaceship originally spinning?

iii) What is the time constant λ in terms of the parameters: a, b, c, T ?