Classical Mechanics Qualifier (January 2012)
George Mason University

You will have **THREE** hours to complete the exam.
You are allowed to use your graduate textbook during the exam.

Short Problems – do both (20 points each):

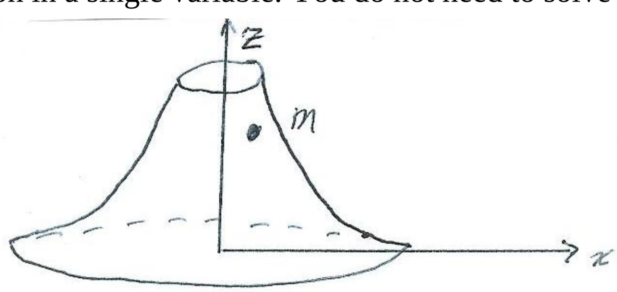
Problem 1

A mass m is constrained to slide frictionlessly on the surface obtained by rotating the curve $z = \frac{A}{x}$ around the z axis. A is a constant greater than zero. Gravity is in the negative z direction.

a) Using the method of Lagrangian multipliers, write down the Lagrangian for the mass using the cylindrical coordinates (r, θ, z) and find the equations of motion.

b) Identify any cyclic coordinates. What is the corresponding conserved quantity?

c) Eliminate the constraints and the cyclic coordinate(s) to obtain a single equation of motion in a single variable. You do not need to solve this equation.

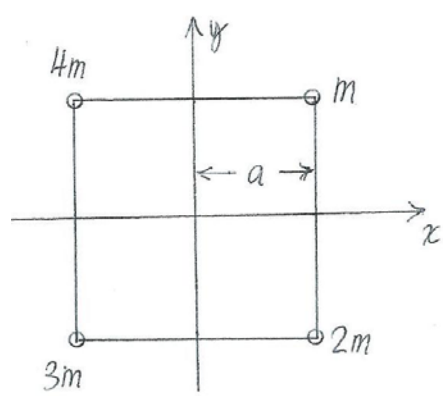


Problem 2

Consider the rigid object formed by the four point masses (m, $2m$, $3m$, and $4m$) placed at the corners of a square of side $2a$, as shown.

a) Find the moment of inertia tensor of the system with respect to the origin (*not* with respect to the center of mass).

b) Find the principal axes and principal moments of inertia, again with respect to the origin.



Long Problems – do two of the following three (30 points each):

Problem 3

A physical system with two degrees of freedom is described by the following Hamiltonian,

$$H = \left(\frac{p_1 - p_2}{2q_1}\right)^2 + p_2 - (q_1 + q_2)^2$$

a) Are there any constants of motion for the system described by this Hamiltonian?

b) Let $f_1(q_1, q_2)$, $f_2(q_1, q_2)$, and $g(q_1, q_2)$ be three smooth functions of the generalized coordinates (q_1, q_2). Use the following generating function,

$$F_2(q_1, q_2, P_1, P_2) = \sum_{i=1,2} f_i(q_1, q_2) P_i + g(q_1, q_2)$$

to calculate the resulting canonical transformation in terms of f_1, f_2, and g:

$$Q_1 = Q_1(q_1, q_2, p_1, p_2), \quad P_1 = P_1(q_1, q_2, p_1, p_2)$$
$$Q_2 = Q_2(q_1, q_2, p_1, p_2), \quad P_2 = P_2(q_1, q_2, p_1, p_2)$$

c) Let $f_1(q_1, q_2) = q_1^2$ and $f_2(q_1, q_2) = q_1 + q_2$ and find an expression for $g(q_1, q_2)$ such that Q_1 and Q_2 are cyclic in the transformed Hamiltonian *K*?

d) What are the constants of motion in this transformed Hamiltonian *K*?

e) Solve $Q_1(t)$ and $Q_2(t)$ as functions of time explicitly using Hamilton's equations.

Problem 4

Consider the orbits of a mass *m* in a central inverse-cube force, $F = \frac{-k}{r^3}$, where *k* is a positive constant. Solve the radial equation of motion for *r* as a function of the angle, ϕ. You will need to consider three cases:

a) Large angular momentum: $l > \sqrt{mk}$.

b) Small angular momentum: $l < \sqrt{mk}$.

c) $l = \sqrt{mk}$.

For each case, describe the orbits and state whether they are bound or unbound. Identify any asymptotes of the orbits.

Problem 5

Three beads with equal mass m slide frictionlessly on a thin hoop of radius R. They are connected by springs that wrap around the hoop. The springs all have force constant k. (The springs also slide frictionlessly around the hoop.) When the beads are equally spaced around the hoop, the springs are unstretched. There is no gravity.

a) Write the Lagrangian of the system using the angles θ_1, θ_2, and θ_3, which are measured from equally spaced, but arbitrary, positions around the ring 120° apart.
b) Write down the matrix you would use to find the resonant frequencies.
c) The resonant frequencies are $\omega = 0$ and $\omega = \sqrt{\frac{3k}{mR^2}}$. (You do not need to show this.) What are the corresponding normal modes? (Note: the non-zero frequency is degenerate (a double root). It is sufficient that you find two *independent* eigenvectors for this frequency and they do not need to be orthogonal to each other.)
d) Give a physical interpretation of each of the normal modes found in (c).