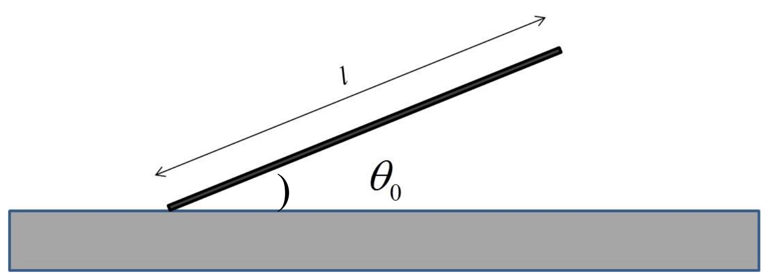
Classical Mechanics Qualifier (August 2012)
George Mason University

You will have **THREE** hours to complete the exam.
You are allowed to use your graduate textbook during the exam.

Short Problems – do both (20 points each):

Problem 1

A uniform thin rod of length l and mass m is initially placed with one end on a frictionless table making an angle θ_0 as shown. i) Write down the Lagrangian for the rod. ii) Determine the time it takes for the rod to fall to the table after it is released from its initial slanted position. (The answer can be left as a definite integral.) iii) How far will the lower end of the rod move horizontally during this time?



Problem 2

A system is described by the Hamiltonian, $H = \frac{p^2}{2} - \frac{1}{2q^2}$. Write down the Hamilton's equation of motion for this system. Show that $F = \frac{pq}{2} - Ht$ is a constant of motion for this system. Here t is time.

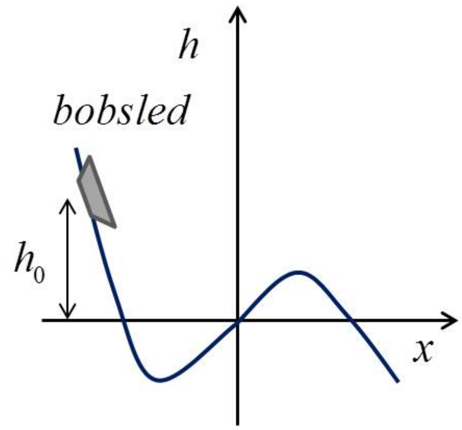
Long Problems – do two of the following three (30 points each):

Problem 3

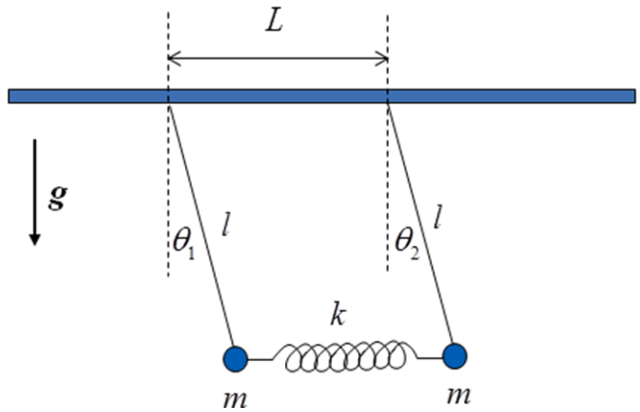
A bobsled is sliding from an initial height h_0 on an ice track with a shape given by a cubic function

$$h(x) = x - \frac{x^3}{3a^2}$$

parameterized by a parameter a as shown. Use the method of the Lagrangian multiplier to find the algebraic equation for the horizontal location x at which the bobsled will lose contact with the track. If the bobsled loses contact with the track at $x = a$, what is the initial height h_0 in terms of the system parameter a?



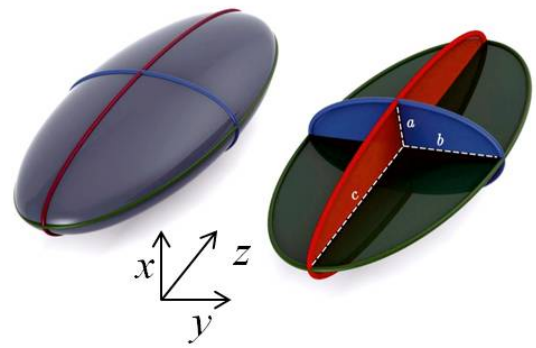
Problem 4



Two pendula with equal mass m and length l are connected by a spring with a spring constant k. The unstretched length of the spring is equal to the distance L between the support points of the two pendula. Considering small oscillations along the line between the two masses (consider only stretching of the spring horizontally and ignore the small vertical stretching), find the eigenfrequencies (resonant frequencies) and the normal modes for the system.

Problem 5

A spaceship lost its power in deep space (far away from any stars). The spaceship was originally spinning along one of its principal axes with a period T. A small perturbation to the angular velocity was seen to grow exponentially in time, $\delta\omega(t) \sim e^{\lambda t}$. Assume the spaceship to have a uniform density ρ, a total mass M, and an ellipsoidal shape with three semi-principal axes $a < b < c$ (see figure).



i) Calculate the Principal Moment of Inertia for the ellipsoid with respect to its center of mass. [Hint: If one rescales the axes, $x = au, y = bv, z = cw$, one can change the ellipsoid to a unit sphere for the integration, i.e.,

$$V = \int_{ellipsoid} dxdydz = \int_{sphere} (abc)dudvdw = \frac{4\pi}{3}abc \quad]$$

ii) Around which axis $(\hat{x}, \hat{y}, \hat{z})$ was the spaceship originally spinning?

iii) What is the time constant λ in terms of the parameters: a, b, c, T?