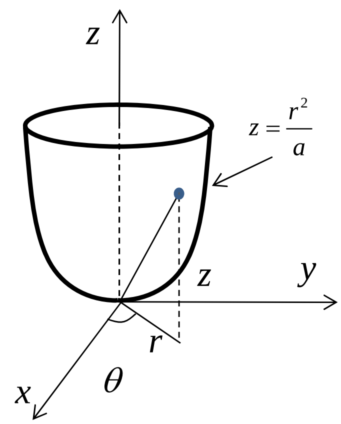
Classical Mechanics Qualifier (Fall 2010)
George Mason University

You will have **THREE** hours to complete the exam. Choose **three** out of the following **four** problems. You are allowed to use your graduate textbook during the exam.

Problem 1 (20pts)
A particle of mass *m* is constrained to move under the influence of gravity on the inside of a smooth parabolic surface of revolution given by $r^2 = az$. Use the Lagrange undetermined multiplier method to derive the constraint force for this problem. Write your answer as a vector in cylindrical coordinates. (Hint: You might want to use the two constants of motion *E* and *l* to simplify some of your expressions. The magnitude of the constraint force is proportional to $\left(1+\frac{4r^2}{a^2}\right)^{-3/2}$.)



Problem 2 (20pts)
A distant star is surrounded by a dust cloud. A planet at a distance *r* away moves under the influence of the star with the familiar inverse-square potential $V_0(r) = -k/r$ and the dust cloud contributes a small additional factor $V'(r) = ar^2/2$. The planet is observed to revolve around the star in a nearly circular orbit with an average radius r_0.

a) For the given central force potential

$$V(r) = V_0(r) + V'(r) = -\frac{k}{r} + \frac{1}{2}ar^2$$

show that the radius for the circular orbit r_0 is given by the following expression:

$$r_0\left(k + ar_0^3\right) = \frac{l^2}{m}$$

where m is the reduced mass of the system and l is the angular momentum of the system.

b) Consider the observed orbit as a small deviation from this circular orbit, show that the apsides will advance approximately by $\frac{3a\pi}{m\omega_0^2}$ per revolution, where ω_0 is the angular frequency for the circular orbit.

Problem 3 (20pts)

Let I_1, I_2, I_3 be the three principal moments of inertia relative to the center of mass of a rigid body and suppose that all these moments are different and they are ranked according to $I_1 > I_2 > I_3$. The rigid body is set to spin around one of its principle axes in free space (with no external force) with an angular velocity $\boldsymbol{\omega}$. Show that the motion is stable if the object is spinning about the principal axes corresponding to I_1 and I_3 (the largest and the smallest moments of inertia) and unstable about the principal axis corresponding to I_2. Explain this analytically using the Euler's equations.

Problem 4 (20pts)

A particle of mass *m* described by one generalized coordinate *q* moves under the influence of a potential *V*(*q*) and a damping force $-2m\gamma\dot{q}$ proportional to its velocity.

a) Show that the following Lagrangian gives the desired equation of motion.

$$L = e^{2\gamma t}\left(\frac{1}{2}m\dot{q}^2 - V(q)\right)$$

b) Obtain the Hamiltonian *H*(*q*,*p*,*t*) for this system.

c) Consider the following generating function:

$$F(p,q,Q,P,t) = e^{\gamma t}qP - QP$$

obtain the canonical transformation from (*q*, *p*) to (*Q*, *P*) and the transformed Hamiltonian *K*(*Q*, *P*, *t*).

d) Pick $V(q) = \frac{1}{2}m\omega^2 q^2$ as a harmonic potential with a natural frequency ω. Show that the transformed Hamiltonian yields a constant of motion

$$K = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 Q^2 + \gamma QP$$

e) Obtain the solution *Q*(*t*) for the damped oscillator in the under-damped case $\gamma < \omega$ by solving Hamilton's equations in the transformed coordinates. Then, write down the solution *q*(*t*) using the canonical transformation obtained in part c.