Qualifying exam - January 2014

Statistical Mechanics

You can use one textbook. Please write legibly and show all steps of your derivations. Note the Formula Sheet attached.

Problem 1 [26 points]

Consider a gas of identical classical particles, each of mass m, in thermal equilibrium at a temperature T. If $\mathbf{v} = (v_x, v_y, v_z)$ is the particle velocity and v is its speed, calculate the following average values:

(a) [2 points] $\overline{v_x}$

(b) [4 points] $\overline{v_x^2}$

(c) [2 points] $\overline{v_x^3}$

(d) [4 points] $\overline{v^2 v_x}$

(e) [4 points] $\overline{v_x^2 v_y^2}$

(f) [4 points] $\overline{v}$

(g) [4 points] $\overline{(1/v)}$

(h) [2 points] Using the results of (f) and (g), show that the general inequality $\bar{v}\overline{(1/v)} > 1$ appearing in Problem 5 is satisfied.

Problem 2 [24 points]

Consider a free electron gas at $T = 0$ K. Suppose its volume is V and the number of electrons is N.

1. [4 points] Show that the total kinetic energy of the gas is

$$U_0 = \frac{3}{5} N \varepsilon_F, \tag{1}$$

where ε_F is the Fermi energy.

2. [5 points] Derive the following relation between the gas pressure p and total energy U_0:

$$pV = \frac{2}{3} U_0. \tag{2}$$

3. [5 points] Show that the isothermal compressibility of the gas, $\beta_T = -\left(\partial \ln V / \partial p\right)_{T,N}$, equals

$$\beta_T = \frac{3V}{2N\varepsilon_F}. \tag{3}$$

4. [5 points] The speed of sound in a gas is given by

$$v_s = \left[\left(\partial p / \partial \rho\right)_T\right]^{1/2}, \tag{4}$$

where ρ is the gas density (mass per unit volume). Compute v_s for the free electron gas at $T = 0$ K and compare it with the Fermi velocity v_F.

5. [5 points] If v is the electron speed, calculate $\overline{v}$, $\overline{(1/v)}$, and check if the general inequality $\overline{v}\,\overline{(1/v)} > 1$ appearing in Problem 5 is satisfied.

Problem 3 [24 points]

Imagine a harmonic solid with an isotropic dispersion relation $\omega = Ak^b$, where ω is the angular frequency of atomic vibrations, k is the wave number, and $A > 0$ and $b > 0$ are constants. Assuming that this dispersion relation holds for each of three polarizations of phonons, show that in the low-temperature limit the phonon contribution to the heat capacity of the solid is proportional to $T^{3/b}$.

Problem 4 [26 points]

A system has two quantum states, state 0 with energy 0 and state 1 with energy ε. These states can be occupied by non-interacting fermions from a particle and heat reservoir at a temperature T and chemical potential μ.

1. [6 points] Calculate the grand partition function $\Gamma(T, \mu)$ of the system.
2. Using the obtained $\Gamma(T, \mu)$, compute the following properties as functions of T and μ:
   (a) [6 points] Average occupation numbers of the two states, $\bar{n}_0$ and $\bar{n}_1$.
   (b) [6 points] Average total energy $\bar{E}$.
   (c) [8 points] The system entropy S.

Extra Credit Problem

Problem 5 [10 points]

Prove that for *any* probability distribution of classical or quantum particles

$$\overline{v}\,\overline{(1/v)} > 1, \tag{5}$$

where v is the particle speed.

Formula Sheet

Moments of the Gaussian function:

$$M_n = \int_0^\infty x^n e^{-x^2} dx. \tag{6}$$

Selected values: $M_0 = \sqrt{\pi}/2$, $M_1 = 1/2$, $M_2 = \sqrt{\pi}/4$, $M_3 = 1/2$, $M_4 = 3\sqrt{\pi}/8$, $M_5 = 1$, $M_6 = 15\sqrt{\pi}/16$.