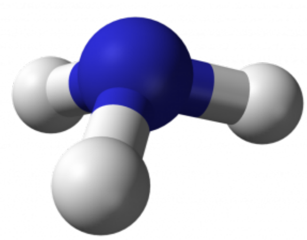
Qualifying exam - August 2017

Statistical Mechanics

You can use one textbook. Please write legibly and show all steps of your derivations. A formula sheet is attached.

Problem 1 [15 points]
The ammonia (NH_3) molecule has the structure of a triangular pyramid, with the N atom in one corner and three hydrogen atoms in other corners, as in the Figure below. Calculate the internal energy (in J/mol) and specific heat at a constant volume (in J/mol/K) of ammonia at the temperature of 1000 K. Consider ammonia as an ideal gas and treat the molecular rotations and vibrations in the classical limit. The gas constant is $R = 8.314$ J/mol/K.



Problem 2 [25 points]
Consider a substance composed of identical particles of a mass m. Using classical statistics, calculate

1. [10 points]

$$\overline{(v - \overline{v})^2}, \tag{1}$$

where v is the magnitude of velocity of the center of mass of the particle.

2. [15 points]

$$\overline{\left(K - \overline{K}\right)^2}, \tag{2}$$

where K is the kinetic energy of the center of mass of the particle.

Problem 3 [30 points]
Consider a cavity containing black-body radiation at a temperature T_1. The Planck formula

$$u_\omega(\omega, T) = \frac{V\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\hbar\omega/kT} - 1}. \tag{3}$$

gives the energy distribution function $u_\omega(\omega, T)$, where ω is the angular frequency of the electromagnetic waves. The plot of $u_\omega(\omega, T_1)$ as a function of ω has a maximum at a frequency ω_1.

Suppose the volume of the cavity increases in an equilibrium adiabatic process from an initial value V_1 to a final value $V_2 = 27V_1$.

1. What is the final temperature T_2 in the cavity? [5 points]
2. If the initial radiation pressure was p_1, what is the final pressure p_2? [5 points]
3. What is the final frequency ω_2 of the maximum of $u_\omega(\omega, T_2)$? [10 points]
4. If the cavity initially contained a total of N_1 photons, what is the final number of photons in the cavity? [10 points]

Problem 4 [30 points]
Consider a quantum gas of ultra-relativistic particles (bosons of fermions) with the energy-momentum relation $\varepsilon = cp$, where c is speed of light. Show that at any temperature

$$PV = \frac{E}{3}, \tag{4}$$

where P is pressure, V is volume of the gas and E is its total energy.

Does this result remain valid for an ultra-relativistic gas in the Maxwell-Boltzmann statistics?

Formula Sheet

Moments of the Gaussian function:

$$M_n = \int_0^\infty x^n e^{-x^2} dx. \tag{5}$$

Selected values: $M_0 = \sqrt{\pi}/2$, $M_1 = 1/2$, $M_2 = \sqrt{\pi}/4$, $M_3 = 1/2$, $M_4 = 3\sqrt{\pi}/8$, $M_5 = 1$, $M_6 = 15\sqrt{\pi}/16$.