# Qualifying exam - August 2014

# Statistical Mechanics

You can use one textbook. Please write legibly and show all steps of your derivations. Note the Formula Sheet attached.

**Problem 1** [20 points]
Consider a substance composed of identical particles of mass $m$. Using classical statistics, calculate the most probable value, $K_m$, of the kinetic energy $K$ of the center of mass of a particle. Compare it with the canonical average kinetic energy $\bar{K}$ of a particle. If the two values are different, explain why.

**Problem 2** [30 points]
Consider a system of localized identical quantum harmonic oscillators with an angular frequency $\omega$. The energy of an oscillator is quantized by $\varepsilon_n = \hbar\omega/2 + n\hbar\omega$, where $n = 0, 1, 2, ....$ The system has been equilibrated with a thermostat at a temperature $T$.

1. [5 points] Calculate the average $\bar{n}$ of the quantum number $n$ as a function of $T$.
2. [10 points] Calculate the root-mean-square fluctuation

   $$\Delta n \equiv \left(\overline{(n-\bar{n})^2}\right)^{1/2}$$

   and the relative fluctuation

   $$v \equiv \frac{\Delta n}{\bar{n}}$$

   as functions of $T$.
3. [5 points] Show that $v > 1$ at any temperature.
4. [10 points] For one of the oscillators, let $\tilde{p}_1$ be the probability of finding it in an excited state with $n > 1$. In other words, if numerous measurements of $n$ have been made, $\tilde{p}_1$ is the fraction of the measurements that gave $n > 1$. Calculate $\tilde{p}_1$ and sketch qualitatively its dependence on $T$. Explain the physical meaning of this plot.

**Problem 3** [30 points]
Consider a gas in equilibrium with a solid surface containing $\nu$ identical adsorption sites per unit area. The energy of an adsorption site is zero if it is unoccupied, $\varepsilon_1$ if singly occupied, and $\varepsilon_2$ if doubly occupied. These energies are independent of whether neighboring adsorption sites are occupied or vacant. The temperature of the system is $T$ and the chemical potential of particles in the gas is $\mu$. Apply the grand canonical formalism to calculate:

1. [10 points] The average number of adsorbed particles per unit area.
2. [10 points] The average energy of the adsorbed particles per unit area.
3. [10 points] The average entropy of the adsorbed particles per unit area.

**Problem 4** [20 points]
Consider a three-dimensional free electron gas at zero temperature (degenerate electron gas). For an arbitrary axis $x$,

1. [10 points] Calculate the mean-square projection $\overline{v_x^2}$ of the electron velocity on $x$.
2. [10 points] Calculate average speed $\overline{v_\perp}$ of the electrons in the plane normal to $x$.

Express your answers in terms of the Fermi energy $\varepsilon_F$ and electron mass $m$.

*Please note the Formula Sheet attached*

## Formula Sheet

Moments of the Gaussian function:

$$M_n = \int_0^\infty x^n e^{-x^2} dx. \tag{1}$$

Selected values: $M_0 = \sqrt{\pi}/2$, $M_1 = 1/2$, $M_2 = \sqrt{\pi}/4$, $M_3 = 1/2$, $M_4 = 3\sqrt{\pi}/8$, $M_5 = 1$, $M_6 = 15\sqrt{\pi}/16$.