Qualifying exam - August 2013

Statistical Mechanics

You can use one textbook. Please write legibly and show all steps of your derivations. Note the Formula Sheet attached.

Problem 1 [20 points]

Consider a system of non-interacting identical localized oscillators. Using the classical Hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2, \tag{1}$$

(m is the particle mass and x displacement from equilibrium) calculate

1. [10 points]
$$\overline{\left(x^2 - \overline{x^2}\right)^2}. \tag{2}$$
2. [10 points]
$$\overline{\left(p^2 - \overline{p^2}\right)^2}. \tag{3}$$

Problem 2 [20 points]

Calculate the average energy per photon in black-body radiation (total energy divided by the number of photons). Show that this energy is approximately $\varepsilon \approx 2.701kT$.

Problem 3 [30 points]

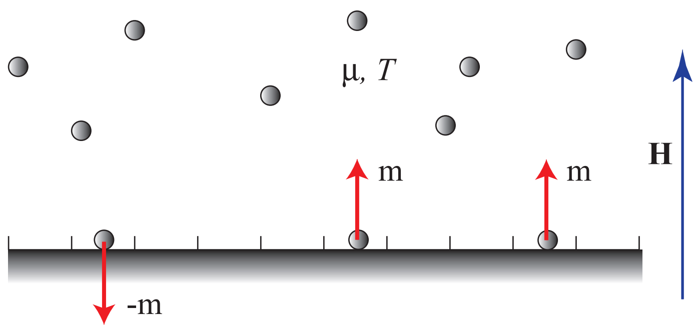
Two-dimensional universe! Imagine that our universe is two-dimensional (2D). By analogy with the 3D theory of black-body radiation, develop a similar theory for a 2D universe. Specifically, consider a cavity of an area (2D "volume") A filled with black-body radiation at a temperature T . Derive the following thermodynamic properties:

1. [5 points] Helmholtz free energy $F(T, A)$.
2. [3 points] Entropy $S(T, A)$.
3. [3 points] Radiation pressure $p(T)$.
4. [4 points] Energy $E(T, A)$. Is the Stefan-Boltzmann Law still valid?
5. [4 points] Specific heat $C_v(T, A)$.
6. [5 points] Total number of photons $N(T, A)$.
7. [6 points] Fundamental equation of state $S(E, A)$.

Problem 4 [30 points]

Consider a gas in equilibrium with a solid surface containing identical adsorption sites. When a molecule adsorbs, its energy changes by $\varepsilon < 0$ due to chemical interaction with the surface. In addition, it acquires a magnetic moment m which can be aligned either parallel or anti-parallel to an applied magnetic field H. Interaction between the adsorbed molecules can be neglected. For given temperature T and chemical potential μ in the gas, apply the grand-canonical formalism to

1. [8 points] Calculate the average fraction of surface sites occupied by molecules.
2. [7 points] Calculate the average magnetic moment $\overline{m}$ per surface site.
3. Now consider the small-field limit, i.e., $mH \ll kT$ at fixed values of ε and μ.
   3a.[7 points] Show that the magnetic moment of the surface is proportional to H.
   3b.[8 points] Find the mean-squared fluctuation $\overline{(m - \overline{m})^2}$.



Formula Sheet

Riemann's zeta function:

$$\varsigma(n) = \frac{1}{(n-1)!} \int_0^\infty \frac{x^{n-1}}{e^x - 1} dx. \tag{4}$$

Selected values: $\varsigma(2) = \pi^2/6$, $\varsigma(3) \approx 1.202$ and $\varsigma(4) = \pi^4/90$.

Moments of the Gaussian function:

$$M_n = \int_0^\infty x^n e^{-x^2} dx. \tag{5}$$

Selected values: $M_0 = \sqrt{\pi}/2$, $M_1 = 1/2$, $M_2 = \sqrt{\pi}/4$, $M_3 = 1/2$, $M_4 = 3\sqrt{\pi}/8$, $M_5 = 1$, $M_6 = 15\sqrt{\pi}/16$.