# Statistical Mechanics Qualifying Exam

Spring 2026

January 14 (1:00 pm - 4:00 pm)

1. [Hint: This entire problem requires no calculation!]

(a) An Ising magnet has spins $S_i$ (local magnetic moments) which can point only in two different directions, i.e. $S_i = \pm 1$. The spins live on a lattice with $N$ sites, labeled as $i = 1, \dots, N$. Suppose that the lattice is a one-dimensional chain. Then, we can use the site index $i$ as a discrete coordinate along the chain and express the Hamiltonian (total energy) of the magnet as:

$$H = -J \sum_i S_i \cdot S_{i+1}$$

Assume $J > 0$, i.e. a ferromagnetic coupling between the nearest-neighbor spins. What is the exact entropy of this magnet at (i) zero temperature and (ii) infinite temperature?

(b) A Potts model is similar to the Ising model, but its "spins" can have $p > 2$ different states (Ising model is a $p = 2$ Potts model). Suppose that Potts spins live on a one-dimensional chain with $N$ sites, and that their Hamiltonian is

$$H = -J' \sum_i \delta_{S_i, S_{i+1}}$$

so that energy is locally gained ($J' > 0$) only when the nearest-neighbor spins are in the same state. What is the exact entropy of this system at (i) zero temperature and (ii) infinite temperature?

(c) Finally, consider a $p$-state Potts model on the (two-dimensional) square lattice with $N$ sites, and let the spins interact with the same type of interaction

$$H = -\sum_{ij} J'_{ij} \delta_{S_i, S_j}$$

but only if they are next-nearest neighbors on the lattice. In other words, $J'_{ij} \neq 0$ only if it takes exactly two steps across lattice links to go from the site $i$ to the site $j$ (and not return to the same site). What is the exact entropy of this system at (i) zero temperature and (ii) infinite temperature?

2. A soap bubble of radius $R$ traps air at pressure $p$, which is higher than the external ambient pressure $p_0$. The pressure difference is balanced by the surface tension that stretches the bubble – by definition, this is a "two-dimensional" pressure inside the soap film, i.e. the tangential force $F_t$ per unit-length:

$$\sigma = \frac{dF_t}{dl}$$

The equation of state for the soap bubble can be approximated as

$$A = 4\pi R^2 = A_0 - b\sigma$$

in the relevant temperature and pressure range, where $A$ is the bubble's area and $A_0, b > 0$ are constants with appropriate units. The air inside the bubble has a fixed number of particles $N$ and its equation of state is

$$pV = Nk_BT$$

(a) [OPTIONAL for extra credit] Show that

$$\sigma = \frac{R}{2}(p - p_0)$$

The remaining questions can be answered by using this formula without proof, and even without understanding what $\sigma$ means (just treat it as a state variable).

(b) Construct the equation for the radius of the bubble as a function of the external pressure $p_0$, temperature $T$ and the number of trapped air molecules $N$. Do not attempt to solve this equation (it's a complicated 4th order polynomial equation).

(c) Find the isothermal compressibility of the bubble (with its trapped air) as a function of $R, N$ and $T$ from

$$\kappa_T = -\frac{1}{V}\frac{dV}{dp_0}\bigg|_T$$

where $V$ is the volume of the bubble and $p_0$ is the *external* pressure. [Note: keeping the dependence on $R$ is much easier than looking for the more "natural" expression $\kappa_T(p_0, T; N)$, because it does not require solving the equation from part (b).]

(d) Express $\kappa_T$ as a function of $R, N$ and $p_0$, i.e. eliminate $T$ from part (c) in favor of $p_0$.

3. Consider the ideal non-relativistic gas at temperature $T$ in general $d$ dimensions. The particles have mass $m$ and no internal degrees of freedom.

(a) Find the most probable speed $v_0$ of a particle.

(b) Find the average speed $\langle v \rangle$ of the particles. Express all dependence on $d$ using Gamma functions (which need not be further simplified).

4. What is the compressibility of the ideal quantum gas of non-relativistic spin-$S$ fermions at zero temperature (in three dimensions)? Specifically:

(a) isothermal compressibility if the number of particles $N$ is kept fixed?

(b) isothermal compressibility if the chemical potential $\mu$ is kept fixed?

(c) compressibility at constant entropy $S$ and fixed number of particles $N$?

Express all results in terms of the particle concentration $n = N/V$, mass $m$, and fundamental constants. You may leave the symbol for Fermi energy $\epsilon_F$ in the final formulas for simplicity if you separately provide a formula for $\epsilon_F$ in terms of $n$, $m$ and constants.

Useful formulas:

- Gamma function:

$$\Gamma(x) = \int_0^\infty dt\, t^{x-1} e^{-t}$$

- Integration of scalars in $d$ dimensions ($\Omega_d$ is the solid angle):

$$\int d^d x\, f(|\mathbf{x}|) = \Omega_d \int_0^\infty dx\, x^{d-1} f(x) \quad , \quad \Omega_d = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} = \begin{cases} 2 & , \quad d=1 \\ 2\pi & , \quad d=2 \\ 4\pi & , \quad d=3 \\ \vdots & , \quad \vdots \end{cases}$$

- Thermal expansion coefficient $\alpha$ and compressibility $\kappa$:

$$\alpha = \frac{1}{V}\frac{dV}{dT} \quad , \quad \kappa = -\frac{1}{V}\frac{dV}{dp}$$

- Energy levels of a free quantum particle in a cubic box of side length $L$:

$$\epsilon_{\mathbf{n}} = \frac{\pi^2 \hbar^2}{2mL^2}(n_x^2 + n_y^2 + n_z^2) \quad , \quad n_x, n_y, n_z \in \{1, 2, 3, \dots\}$$