Statistical Mechanics Qualifying Exam

Spring 2025

January 15 (1:00 pm - 4:00 pm)

1. If the particles of an ideal gas have energy $H = |\mathbf{p}|^2/2m$ at momentum $\mathbf{p}$, we know that the equipartition theorem applies, i.e. the total internal energy of all particles at temperature T is $E = \mathcal{N} \cdot k_B T/2$, where $\mathcal{N}$ is the total number of degrees of freedom. Each direction in which a particle can move is a degree of freedom, and every particle has its own degrees of freedom, so N particles in three dimensions give $\mathcal{N} = 3N$.

   (a) Does the equipartition theorem hold in the same form when the energy of a particle is $H = \lambda(p_x^4 + p_y^4 + p_z^4)$? If it does not, derive the internal energy $E(N, T)$.

   (b) Does the equipartition theorem hold in the same form when the energy of a particle is $H = a|\mathbf{p}|^4$? If it does not, derive the internal energy $E(N, T)$.

2. A mixture of 0.1 mole of helium ($\gamma_1 = C_p/C_v = 5/3$) with 0.2 mole of nitrogen ($\gamma_2 = 7/5$), considered an ideal mixture of two ideal gases, is initially at 300 K and occupies a volume of 4 liters. Show that the changes of temperature and pressure of the system which occur when the gas is compressed slowly and adiabatically can be described in terms of some intermediate value of γ. Calculate the magnitude of these changes when the volume is reduced by 1%.

3. A dielectric consists of N localized atoms. An undisturbed atom can either be completely neutral with zero energy and no dipole moment, or polarized into a dipole with a fixed dipole moment magnitude $p = |\mathbf{p}|$ and energy $\epsilon > 0$. When polarized, an atomic dipole experiences a strong uniaxial anisotropy, so that its dipole moment $\mathbf{p}$ can point only in the $+z$ or $-z$ direction. This dielectric is in equilibrium at temperature T and subjected to an external electric field $\mathcal{E}$ along the z direction.

   (a) Label the possible states of an atom i as $n_i = 0$ if the atom is not polarized, and $n_i = \pm 1$ if the atom is polarized in the $\pm z$ direction. Construct the expression for the Hamiltonian of N atoms in terms of the n_i variables. Recall that a single dipole with moment $\mathbf{p}$ in the electric field $\boldsymbol{\mathcal{E}}$ also has electrostatic potential energy $-\mathbf{p}\boldsymbol{\mathcal{E}}$.

   (b) How many microstates of N atoms with a total internal energy E are there? You do not need to calculate the entropy.

   (c) Use the canonical ensemble to calculate the internal energy of N atoms in thermal equilibrium.

   (d) Use the canonical ensemble to calculate the total polarization P of N atoms in thermal equilibrium.

4. A gas of N non-relativistic spinless bosonic particles of mass m is enclosed in a volume V at temperature T.

(a) Find the density of single-particle states $\rho(\epsilon)$ as a function of the single-particle energy ϵ. Recall that the density of states $\rho(\epsilon)$ gives the number of single-particle quantum states $dN = V\rho(\epsilon)d\epsilon$ in the volume V and the infinitesimal energy interval $(\epsilon, \epsilon + d\epsilon)$.

(b) Write the expression for the mean occupation number of a single particle state $n(\epsilon)$ as a function of the particle's energy ϵ, temperature T and chemical potential μ.

(c) Write an integral expression which implicitly determines $\mu(T)$ from the particle concentration N/V.

(d) Find the Bose-Einstein transition temperature T_c below which one must have a macroscopic occupation of some single-particle states. Leave your answer in terms of a dimensionless integral.

(e) What is $\mu(T)$ for $T < T_c$? Describe the energy dependence of $n(\epsilon)$ for $T < T_c$, either in accurate words or formulas (make sure to point out the main qualitative features that are specific for $T < T_c$).

(f) Find an exact expression for the total energy, $E(T, V)$ of the gas at $T < T_c$. Leave your answer in terms of a dimensionless integral.

Useful formulas:

- Gamma function:

$$\Gamma(x) = \int_0^\infty dt\, t^{x-1} e^{-t} \quad ; \quad \Gamma(x+1) = x\Gamma(x)$$

- Energy levels of a free quantum particle in a cubic box of side length L:

$$\epsilon_{\mathbf{k}} = \frac{\hbar^2 |\mathbf{k}|^2}{2m} \quad , \quad \mathbf{k} = \frac{\pi}{L}(\hat{\mathbf{x}} n_x + \hat{\mathbf{y}} n_y + \hat{\mathbf{z}} n_z) \quad , \quad n_x, n_y, n_z \in \{1, 2, 3, \dots\}$$

- Summing over discrete quantum numbers $n_x, n_y, n_z \in \{1, 2, 3, \dots\}$ in the limit of infinite volume $V = L^3$:

$$\sum_{\mathbf{n}} \to \int_0^\infty dn_x \int_0^\infty dn_y \int_0^\infty dn_z = \left(\frac{L}{\pi}\right)^3 \int_0^\infty dk_x \int_0^\infty dk_y \int_0^\infty dk_z = \frac{1}{8}\left(\frac{L}{\pi}\right)^3 \int_{-\infty}^\infty dk_x \int_{-\infty}^\infty dk_y \int_{-\infty}^\infty dk_z$$