# Statistical Mechanics Qualifying Exam

Spring 2024

January 10 (1:00 pm - 4:00 pm)

1. Non-interacting particles of mass $m$ are kept in equilibrium at temperature $T$. The container which holds the gas of these particles is tall enough that the effects of gravity cannot be neglected; the density $\rho(h)$ of the gas is not constant as a function of the height $h$. Determine the density ratio $\rho(h_1)/\rho(h_2)$ at any two different heights $h_1, h_2$, *using kinetic theory of gases*:

   (a) Let $f(\mathbf{r},\mathbf{p})$ be the probability distribution function of the gas, such that $f(\mathbf{r},\mathbf{p})\, d^3r\, d^3p$ is the probability for a particle to be within the spatial volume $d^3r$ and momentum-space volume $d^3p$ surrounding the point $(\mathbf{r},\mathbf{p})$ in the phase space. Write the expression for $f(\mathbf{r},\mathbf{p})$ in the canonical ensemble, up to a normalization constant which you don't need to calculate. Take into account both the kinetic and potential energy of the particles.

   (b) Express the local density $\rho(\mathbf{r})$ of the gas in terms of $f(\mathbf{r},\mathbf{p})$. Using this, find the ratio $\rho(h_1)/\rho(h_2)$. [Note: the final result is very simple and easy to obtain without calculation, so in order to get full credit you must explain and justify your steps.]

2. An elastic rubber band has a negligible thermal expansion coefficient, and hence a fixed length $L_0$ at any temperature when no tension force is applied to it. It is experimentally found that the tension $f$ increases in proportion to the temperature increase if the band's length $L$ is fixed, so that the internal energy has the temperature dependence

$$E = nc(L)T$$

   where $n$ is the fixed number of molecules in the band and $c(L)$ is a length-dependent proportionality constant. Furthermore, the band obeys Hooke's law

$$f = \frac{\phi(T)}{n}(L - L_0)$$

   i.e. the force of tension $f$ at any fixed temperature $T$ is proportional to the change of the band's length $L - L_0$.

   (a) Write the 1st law of thermodynamics for this system, relating the infinitesimal changes of internal energy $E$ to the changes of entropy $S$ and length $L$. Then, assuming $L = \text{const.}$, express the heat intake in the 1st law using $E = ncT$ (and recall that $c$ does not depend on temperature). Also, substitute Hooke's law. At this point, derive the partial derivatives of the internal energy with respect to temperature and length

$$\left.\frac{dE}{dT}\right|_L \quad , \quad \left.\frac{dE}{dL}\right|_T$$

   in terms of $c(L)$, $\phi(T)$ and $n, L, L_0$.

   (b) Starting from the last result, derive the expressions for $\phi(T)$ and $c(L)$. Manipulate the energy derivatives from part (a) in order to eliminate $E$ and construct a relationship between $d\phi/dT$ and $dc/dL$. Then integrate out the temperature to obtain $\phi(T)$. The expression for $\phi(T)$ you get must not depend on $L$, so you can deduce the formula for $c(L)$. Ultimately show that $\phi(T)$ is proportional to temperature $T$. Verify your formulas for $\phi(T)$ and $c(L)$ by plugging them into the 1st law and reproducing

$$E(L,T) = nc(L)T$$

(c) Derive the expression for entropy change $S(T) - S(T_0)$ with the change of temperature from $T_0$ to $T$ at constant $L$.

(d) Find the change of the band's length due to temperature increase, $dL/dT$, at a constant tension force $f$. Is the result consistent with usual experience regarding thermal expansion? If yes, explain what causes the usual experience. If not, explain which stated property of this system is unusual and responsible for the unconventional behavior.

3. The three lowest energy levels of a certain molecule are $E_1 = 0$, $E_2 = \epsilon$ and $E_3 = 10\epsilon$. Show that at sufficiently low temperatures $T < T_c$ only levels $E_1$ and $E_2$ are populated. Estimate $T_c$ for a gas of $N$ molecules by the condition that the number of molecules in the state $E_3$ is $N_3(T = T_c) \sim 1$. Find the average energy $\langle E \rangle$ of a molecule at temperature $T$. Find the specific heat $C$, identify its low-temperature and high-temperature behaviors, then sketch $C(T)$.

4. Consider a quantum ideal gas of non-relativistic spin $S = \frac{1}{2}$ fermions with mass $m$ in two dimensions.

(a) Find the density of states $\rho(\epsilon)$ assuming the usual energy dispersion

$$\epsilon = \frac{\pi^2\hbar^2}{2mL^2}(n_x^2 + n_y^2) \quad , \quad n_x, n_y \in \{1, 2, 3, \ldots\}$$

in a square box of side-length $L$. Show that $\rho(\epsilon)$ is constant, i.e. not dependent on energy in two dimensions. [DOS is the number of single-particle quantum states per unit volume (area), per unit energy interval].

(b) The particle concentration $n$ is given by

$$n = \int_0^\infty d\epsilon\, \rho(\epsilon) n(\epsilon) \quad , \quad n(\epsilon) = \frac{1}{z^{-1}e^{\beta\epsilon} + 1}$$

where $n(\epsilon)$ is the fermionic "occupation number" (the average number of fermionic particles in a quantum state with energy $\epsilon$), and $z = e^{\beta\mu}$ is fugacity determined by the chemical potential $\mu$ and inverse temperature $\beta = 1/k_B T$. Solve the given integral to find $n(\mu, T)$. For simplicity, don't substitute the detailed expression for $\rho(\epsilon) \equiv \rho = \text{const}$. Then, obtain the exact formula for the chemical potential $\mu(n, T)$ as a function of concentration and temperature.

(c) Show that $\mu(T)$ approaches a finite value when $T \to 0$, with a correction that vanishes as an exponential function of temperature. This finite value is the Fermi energy, $\mu(T = 0) = \epsilon_F$. What is the temperature dependence of $\mu(T)$ in the high-temperature limit?

(d) [OPTIONAL for extra credit] Derive the equation of state which relates the pressure $p$ to the particle concentration and temperature. First show that the pressure of non-relativistic particles in two-dimensions is equivalent to energy per unit volume (area). Then, construct the integral formula for energy density $\mathcal{E} = E/V$ analogous to the expression for $n = N/V$ given in part (b). Try to simplify the integral, but do not attempt to solve it (there is no closed form). Instead, obtain the low and high temperature approximations. Show that pressure saturates to a constant (degeneracy pressure) at $T \to 0$, and obeys the classical equation of state in the high-temperature limit.