Qualifying exam-Spring 2023

Statistical Mechanics

You can use one textbook. Please write legibly and show all steps of your derivations.

Problem 1 [20 points]**:**

A resistor with resistance R is held at a constant temperature T. Current I is passed through the resistor for time interval Δt.

(a) [5 points] What is the change in the entropy of the resistor?
(b) [5 points] What is the change in the entropy of the universe?
(c) [5 points] What is the change in the internal energy of the universe?
(d) [5 points] What is the change in the Helmholtz free energy of the universe?

Problem 2 [30 points]**:**

(a) [5 points] Derive the following Maxwell relation:

$$(\frac{\partial S}{\partial V})_T = (\frac{\partial p}{\partial T})_V$$

(b) [5 points] Maxwell found that the electromagnetic radiation (photon gas) in an evacuated vessel of volume *V,* has a pressure which is equal to 1/3 of the energy density: $P = (1/3)\ u(T)=U(T)/3V$. Using the relation obtained in part (a) combined with 1th and 2th laws of thermodynamics prove that *u(T)* satisfies the following equation:

$$u = \frac{T du}{3 dT} - \frac{1 u}{3}$$

(c) [5 points] Solve this equation for u to obtain Stephan-Boltzmann's law.
(d) [10 points] In the big-bang theory, the radiation energy which is initially confined in a small region adiabatically expands in a spherically symmetric way. Based on thermodynamic considerations together with results from part (b), find a relation between the temperature and the radius of the volume of radiation. How does the temperature changes as the radiation expands?
(e) [5 points] Find the entropy of the electromagnetic radiation as a function of T and V.

Problem 3 [15 points]**:**

The entropy of a paramagnet in the presence of an applied magnetic field is given by:
$S = S_0 - CU^2$,
where U is the energy of the system and C is a constant.

(a) (5 points) Find the energy U of the system as a function of temperature T.
(b) (5 points) Sketch U versus T for all values of T ($-\infty < T < \infty$) assuming C>0.
(c) (5 points) Describe the physical interpretation of the negative temperature in part (b).

Problem 4 [15 points]**:**

Consider a 1D chain consisting of n segments (Figure 1) where $n \gg 1$. The length of each segment is a when the long dimension of the segment is parallel to the chain and zero when it is vertical (Each segment has just two states, a horizontal and a vertical one). The distance between the chain ends is nx.

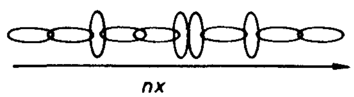

Figure 1

(a) (5 points) Find the entropy of the chain as a function of a and x.
(b) (5 points) Derive a relation between the tension F necessary to maintain the distance nx and the temperature T of the chain assuming the joints turn freely. *Hint: You may want to find the mean length of a segment.*
(c) (5 points) Show that at high temperatures your answer in part (b) leads to Hook's law.

Problem 5 [20 points]:

Consider a system of two non-interacting particle in a canonical ensemble at temperature T. Each particle can be in 5 different states with energies $E = n\varepsilon$, where n=0,1, 2, 3, 4.

(a) [5 points] Compute the partition function in a classical Maxwell-Boltzmann approximation.
(b) [5 points] Compute the partition function assuming Fermi-Dirac statistics.
(c) [5 points] Compute the partition function assuming Bose-Einstein statistics.
(d) [5 points] Compare these partition functions in the limit of high temperatures ($k_B T \gg \varepsilon$) and low temperatures ($k_B T \ll \varepsilon$).

Mathematical Formulas:

$$\sum_{0}^{\infty} x^n = \frac{1}{1-x}$$

$$\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$$