Qualifying exam-Spring 2022

Statistical Mechanics

You can use one textbook. Please write legibly and show all steps of your derivations.

Problem 1 [20 points].

Based on Bekenstein and Hawking theory, the entropy of a black hole is proportional to its area A, and is given by:

$$S=\frac{k_B c^3}{4G\hbar}A,$$

Where the relation between the radius and mass of a black hole is given by

$$R=\frac{2G}{c^2}M,$$

a) (10 points) How the entropy changes when two black holes collapse into one?

b) (10 point) The internal energy of the black hole is given by Einstein relation, $E=Mc^2$. Find the temperature of the black hole in terms of its mass.

Problem 2 [25 points]**:**

A lattice contains N normal lattice sites and N interstitial lattice sites (places between the lattice points where atoms can reside). M identical atoms sit on the interstitial sites and N-M on the normal sites ($N \gg M \gg 1$). If an atom sits on normal sites, its energy $E = 0$. If an atom occupies an interstitial site, its energy is $E = \varepsilon$.

(a) (15 points) What is entropy?

(b) (15 points) Compute the internal energy and heat capacity.

Problem 3 [25 points]
Consider a gas of spinless particles inside a container with volume of 1 cubic meter and in contact with a heat reservoir at temperature T. Assuming the classical Hamiltonian

$$H=\frac{p^2}{2m}$$

Where m is the particle mass, calculate:

(a) (10 points) One particle partition function Z.

(b) (20 points) The energy fluctuations per particle:

$$\overline{(E-\overline{E})^2}=\overline{E^2}-\overline{E}^2$$

Problem 4 [20 points]:

Consider a system of two non-interacting particle in a canonical ensemble at temperature T. Each particle can be in 5 different states with energies $E = n\varepsilon$, where n=0,1, 2, 3, 4.

(a) [5 points] Compute the partition function in a classical Maxwell-Boltzmann approximation.

(b) [5 points] Compute the partition function assuming Fermi-Dirac statistics.

(c) [5 points] Compute the partition function assuming Bose-Einstein statistics.

(d) [5 points] Compare these partition functions in the limit of high temperatures ($k_B T \gg \varepsilon$) and low temperatures ($k_B T \ll \varepsilon$).

Mathematical Formulas:

$$\sum_0^{\infty} x^n = \frac{1}{1-x}$$

$$\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$$