Qualifying exam - January 2018

Statistical Mechanics

You can use one textbook. Please write legibly and show all steps of your derivations. A formula sheet is attached.

Problem 1 [15 points]
Prove that the following thermodynamic relations are valid for any single-component fluid:

(a) [5 points]

$$\left(\frac{\partial U}{\partial V}\right)_{T,N} = T\left(\frac{\partial P}{\partial T}\right)_{V,N} - P \tag{1}$$

(b) [5 points]

$$\left(\frac{\partial p}{\partial T}\right)_{G,N} > 0, \tag{2}$$

(c) [5 points]

$$\left(\frac{\partial p}{\partial T}\right)_{\mu} > 0. \tag{3}$$

Problem 2 [15 points]
A three-dimensional classical oscillator has been equilibrated with a thermostat at a temperature T. The Hamiltonian of the oscillator is

$$H(\mathbf{p},\mathbf{r}) = \frac{p^2}{2m} + a_x x^2 + a_y y^2 + a_z z^2,$$

where a_x, a_y and a_z are constants.

(a) [5 points] Find the expectation values for the kinetic and potential energies of the oscillator.

(b) [10 points] Find the root-mean-square fluctuation of the total energy of the oscillator.

Problem 3 [35 points]
A system has two quantum states, state 0 with energy 0 and state 1 with energy ε. These states can be occupied by non-interacting fermions from a particle and heat reservoir at a temperature T and chemical potential μ.

1. [7 points] Calculate the grand partition function $\Gamma(T,\mu)$ of the system.
2. Using the obtained $\Gamma(T,\mu)$, compute the following properties as functions of T and μ:

(a) [9 points] Average occupation numbers of the two states, $\bar{n}_0$ and $\bar{n}_1$.

(b) [9 points] Average total energy $\bar{E}$.

(c) [10 points] The system entropy S.

Problem 4 [35 points]
Consider a gas of ultra-relativistic spin-s fermions with the energy-momentum relation $\varepsilon = cp$, where c is the speed of light. The gas contains $N \gg 1$ particles occupying a three-dimensional volume V.

(a) [8 points] Calculate the Fermi energy as a function of V and N.

(b) [8 points] Find the average energy per particle at zero temperature.

(c) [9 points] Find the gas pressure P at zero temperature.

(d) [10 points] Find the isothermal compressibility of the gas, $\beta_T = -\left(\partial \ln V / \partial P\right)_{T,N}$, at zero temperature.

Formula Sheet

Moments of the Gaussian function:

$$M_n = \int_0^\infty x^n e^{-x^2} dx. \tag{4}$$

Selected values: $M_0 = \sqrt{\pi}/2$, $M_1 = 1/2$, $M_2 = \sqrt{\pi}/4$, $M_3 = 1/2$, $M_4 = 3\sqrt{\pi}/8$, $M_5 = 1$, $M_6 = 15\sqrt{\pi}/16$.