Qualifying exam - August 2026

Statistical Mechanics

You can use one textbook. No electronic devices are allowed. Please write legibly and show all steps of your derivations. No credit will be given for a solution without a detailed proof. Note the Formula Sheet attached.

Problem 1 [20 points]
Consider a system of localized, identical non-interacting particles each representing a two-level system with energy levels 0 of degeneracy 1 and ε of degeneracy g. The system is equilibrated with a thermostat at a temperature T. Calculate the mean-square fluctuation of the particle energy with respect to its expectation value.

Problem 2 [30 points]
For a classical ideal gas of identical particles, calculate the average speed $\overline{v_\perp}$ of the center of mass of a particle in the plane normal to a given axis x.

Problem 3 [24 points]

Imagine a harmonic solid with an isotropic dispersion relation $\omega = Ak^b$, where ω is the angular frequency of atomic vibrations, k is the wave number, and $A > 0$ and $b > 0$ are constants. Assuming that this dispersion relation holds for each of three polarizations of phonons, show that in the low-temperature limit, the phonon contribution to the heat capacity of the solid is proportional to $T^{3/b}$.

Problem 4 [26 points]

A system has two quantum states, state 0 with energy 0 and state 1 with energy ε. These states can be occupied by non-interacting fermions from a particle and heat reservoir at a temperature T and chemical potential μ.

1. [6 points] Calculate the grand partition function $\Gamma(T, \mu)$ of the system.
2. Using the obtained $\Gamma(T, \mu)$, compute the following properties as functions of T and μ:
   (a) [6 points] Average occupation numbers of the two states, $\bar{n}_0$ and $\bar{n}_1$.
   (b) [6 points] Average total energy $\bar{E}$.
   (c) [8 points] The system entropy S.

Formula Sheet

Moments of the Gaussian function:

$$M_n = \int_0^\infty x^n e^{-x^2} dx. \tag{1}$$

Selected values: $M_0 = \sqrt{\pi}/2$, $M_1 = 1/2$, $M_2 = \sqrt{\pi}/4$, $M_3 = 1/2$, $M_4 = 3\sqrt{\pi}/8$, $M_5 = 1$, $M_6 = 15\sqrt{\pi}/16$.