Statistical Mechanics Qualifying Exam

Fall 2025

August 20 (1:00 pm - 4:00 pm)

1. A black hole is moving through a vast empty region of the universe. If the universe glows with peak intensity at a wavelength λ, what is the equilibrium mass M of the black hole? Assume that the black hole is in thermal equilibrium with the background radiation of the universe, and that it can lose mass by Hawking radiation or gain mass by absorbing ambient radiation. The energy of the black is $E = Mc^2/2$ and its entropy is given by the Bekenstein-Hawking formula

$$S = \frac{k_B A}{4 l_{\mathrm{P}}^2} \quad , \quad l_{\mathrm{P}} = \sqrt{\frac{G\hbar}{c^3}}$$

where A is the event horizon area and l_{P} is Planck's length. Consider only the simplest Schwarzschild black hole, without charge or angular momentum, whose event horizon is a sphere of radius

$$R_s = \frac{2GM}{c^2}$$

2. An amorphous solid droplet is formed from N identical close-packed atoms. Assume that an atom can be in various internal states with k different values of energy and volume, specifically n_i degenerate states with energy ϵ_i and volume v_i for each $i = 1, 2, \ldots, k$. Neglect kinetic energy and empty space between atoms, i.e. approximate the volume of the droplet as the total volume of all atoms.

   (a) Find the thermal expansion coefficient α of the droplet at temperature T. Use canonical ensemble to obtain the general expression for α in terms of temperature, the average volume of an atom $\langle v \rangle$, the average energy of an atom $\langle \epsilon \rangle$, and the energy-volume "correlation" $\langle v\epsilon \rangle$. [If this seems too abstract and technical, try part (b) first.]

   (b) Suppose we want to model alkali atoms in the droplet. There will be two degenerate ground states, $n_1 = 2$, since the valence electron can have two distinct spin projections in its lowest-energy s orbital. Use these ground states as the reference for energy, i.e. set $\epsilon_1 = 0$, and denote the atom's volume in these states as $v_1 = v$. The lowest energy excited states are $n_2 = 6$ p-orbitals, with energy $\epsilon_2 = \Delta > 0$ (due to the partial screening of the nuclear Coulomb potential) and volume $v_2 = xv$. The ratio $x = v_2/v_1$ is larger than 1 because the p-orbitals have angular momentum and thereby a larger radius than the s-orbitals. Find the thermal expansion coefficient in this model with two degenerate energy levels, and obtain its low temperature ($k_B T \ll \Delta$) and high temperature ($k_B T \gg \Delta$) approximations in terms of T, Δ and x.

3. Consider a rubber band of natural length L_0. When a tension force F is applied to it at temperature T, the rubber stretches to a length L that can be deduced from empirically established relations:

$$\frac{\partial F}{\partial L}\Big|_T = \frac{aT}{L_0}\left[1 + 2\left(\frac{L_0}{L}\right)^3\right]$$

$$\frac{\partial F}{\partial T}\Big|_L = \frac{aL}{L_0}\left[1 - \left(\frac{L_0}{L}\right)^3\right]$$

where a is a constant. In the absence of tension, the rubber length is L_0 at any temperature.

(a) Find the equation of state, i.e. F as a function of T, L and constants.

(b) Derive the formula for $(\partial L/\partial T)_F$ and discuss its physical meaning.

4. Find the compressibility of an ideal non-relativistic gas at high temperatures with the lowest-order quantum corrections,

   (a) at constant entropy and number of particles,

   (b) at constant temperature and number of particles.

   Consider both Bose and Fermi gases on an equal footing and give your answer in terms of temperature T, gas concentration n and thermal wavelength $\lambda_T = h/\sqrt{2\pi m k_B T}$.

Useful formulas:

- Chain rule:
$$\frac{da}{db}\Big|_c \frac{\partial b}{\partial c}\Big|_a \frac{\partial c}{\partial a}\Big|_b = -1$$

- Thermal expansion coefficient α and compressibility κ of a gas:
$$\alpha = \frac{1}{V}\frac{dV}{dT} \quad , \quad \kappa = -\frac{1}{V}\frac{dV}{dp}$$

- Energy levels of a free quantum particle in a cubic box of side length L:
$$\epsilon_{\mathbf{n}} = \frac{\pi^2\hbar^2}{2mL^2}(n_x^2 + n_y^2 + n_z^2) \quad , \quad n_x, n_y, n_z \in \{1, 2, 3, \ldots\}$$

- High-temperature expansion for the internal energy of an ideal quantum gas ($\sigma = 1$ is for bosons, $\sigma = -1$ is for fermions, S is the particles' spin):
$$E = \frac{3}{2}Nk_BT\left[1 - \frac{\sigma}{4\sqrt{2}}\left(\frac{n\lambda_T^3}{2S+1}\right) + \mathcal{O}\left((n\lambda_T^3)^2\right)\right]$$