Statistical Mechanics Qualifying Exam

Fall 2024: August 21, 1:00 pm - 4:00 pm

Every problem is worth 25 points.

1. (a) When the gas in a pressurized duster-can flows out through a small hole (nozzle) for a while, the can becomes cold. Explain why this happens (provide a convincing argument which makes use of known physics and thermodynamics laws).

   (b) Would a pressurized *ideal gas* in the can cool down in the previous scenario? Explain.

   (c) Consider a system of N identical, distinguishable, non-interacting atoms. Each atom can be in one of its n internal states. Let us label these internal states by $i \in \{1, 2, \dots, n\}$. The energy ϵ_i of an atom in any internal state i is known. Now, the system is in contact with the environment at temperature T. What is the probability that a particular particle will be in a specific internal state i?

   (d) What is the probability that the system from part (c) will be in the configuration (microstate) with the particle 1 in the state i_1, particle 2 in the state i_2, ..., and particle N in the state i_n?

2. A certain magnetic material has the following equation of state which relates its magnetization M to the applied magnetic field H at temperature T,

$$M(T,H) = Nm\left(1 - \frac{k_B T}{mH}\right) \quad , \quad \frac{k_B T}{mH} \ll 1$$

   It is assumed here that the external field is large, so the magnetization is near saturation ($M \approx mN$). The internal energy of this material is known to depend only on temperature in this regime,

$$E(T,H) = C_M T$$

   where C_M is a constant "heat capacity". The quantities m and N are also constant. The system is taken adiabatically from a state (M_1, T_1) to a state (M_2, T_2). Find the relation between (M_1, T_1) and (M_2, T_2).

   [Hint: The sought relation is analogous to $p_1 V_1^\gamma = p_2 V_2^\gamma$ for the adiabatic process in an ideal gas. In order to find it, consider which thermodynamic state function X stays constant in adiabatic processes. Then, formulate and solve a differential equation for $\delta M(T) = mN - M$, which is the deviation of magnetization from its saturated value $M(T, H \to \infty) \to Nm$. This equation should contain a derivative $d\delta M/dT$ at constant X.]

3. A one-dimensional chain consists of $N \gg 1$ sites arranged horizontally, and every site has a polymer attached to it. A polymer attached to a site can either stretch vertically with zero energy, or tilt in two possible directions (in and out) with an energy cost $\epsilon > 0$ (which does not depend on the tilt direction).

   (a) Find the entropy of this system $S(E, N)$ as a function of its internal energy E and the number of sites N.

(b) Find the energy $E(T,N)$ as a function of temperature T.

(c) Calculate the specific heat C/N and obtain its asymptotic temperature behaviors in the $T \to 0$ and $T \to \infty$ limits.

4. An ideal gas of identical point particles is sealed in a container of fixed volume. The particles have a known mass m and concentration $n = N/V$. Find the energy density $\epsilon = E/V$ as a function of n, T in the following cases:

   (a) At very high temperatures;

   (b) Near absolute zero temperature, assuming that the particles are spinless fermions;

   (c) Near absolute zero temperature, assuming that the particles are spinless bosons.

   (d) Estimate the crossover temperature T_0 above which the quantum exchange statistics of particles becomes irrelevant.

   Feel free to solve this problem in either two or three dimensions, and report results for an infinitely large system. If you answer the question in arbitrary number of dimensions d, you can get extra credit.

Useful formulas:

- The first law of thermodynamics:

$$dE = dQ - pdV + HdM + \cdots$$

  For reversible processes,

$$dQ = TdS$$

- Energy levels of a free quantum particle in a cubic box of side length L:

$$\epsilon_{\mathbf{k}} = \frac{\hbar^2 |\mathbf{k}|^2}{2m} \quad , \quad \mathbf{k} = \frac{\pi}{L}(\hat{\mathbf{x}} n_x + \hat{\mathbf{y}} n_y + \hat{\mathbf{z}} n_z) \quad , \quad n_x, n_y, n_z \in \{1, 2, 3, \dots\}$$

- The density of states: summing over discrete quantum numbers $n_x, n_y, n_z \in \{1, 2, 3, \dots\}$ in the limit of infinite volume $V = L^3$:

$$\sum_{\mathbf{n}} \to \int_0^\infty dn_x \int_0^\infty dn_y \int_0^\infty dn_z = \left(\frac{L}{\pi}\right)^3 \int_0^\infty dk_x \int_0^\infty dk_y \int_0^\infty dk_z = \frac{1}{8}\left(\frac{L}{\pi}\right)^3 \int_{-\infty}^\infty dk_x \int_{-\infty}^\infty dk_y \int_{-\infty}^\infty dk_z$$