Statistical Mechanics Qualifying Exam

Fall 2023

August 16 (1:00 pm - 4:00 pm)

1. (a) Evaluate the root mean square speed $v_{\text{rms}} = \sqrt{\langle v^2 \rangle}$ of a gas obeying Maxwell's distribution in d dimensions. Assume that all molecules of the gas are identical and have mass m.

   (b) Give the numerical value of v_{rms} for air (in $d = 3$ dimensions) at room temperature ($T = 300$ K). Take into account that 2/3 of air moleculs are nitrogen (molecular weight $28m_0$) and 1/3 are oxygen (molecular weight $32m_0$). The atomic mass unit is $m_0 = 1.66 \times 10^{-27}$ kg.

2. A room at temperature T loses heat through its walls to the outside environment at a rate $A(T - T_0)$, where T_0 is the temperature of the environment. The temperature $T > T_0$ in the room is maintained by a heater which runs Carnot cycles (the room and the environment serve as its heat reservoirs). The power consumed by the heater is P.

   (a) What is the maximum rate dQ/dt at which the pump can deliver heat to the room?

   (b) What is the room temperature T as a function of T_0, A and P?

3. Derive the expression for the specific heat $C(T)$ of a diatomic gas due to quantum vibrations at temperature T. The vibrational energy levels of a gas molecule are:

   $$E_n = \hbar\omega \left(n + \frac{1}{2} \right) \quad , \quad n \in \{0, 1, 2, \ldots\}$$

   Assume that there are N molecules in the gas and ignore their motion through space and rotation. Use the canonical ensemble and treat the particles as distinguishable.

   (a) First obtain the partition function.

   (b) Then, calculate the entropy of the gas at temperature T.

   (c) Derive the specific heat from the entropy. Explicitely write the approximate low-temperature and high temperature approximations for the specific heat and comment on its behavior.

4. Consider a quantum mechanical gas of non-interacting spin-zero bosons, each of mass m. Use the quantum grand canonical ensemble. The particles are free to move within a volume V.

   (a) Find the energy and heat capacity at very low temperatures. Discuss why it is appropriate to set the potential enegry to zero in the limit $T \to 0$. Do not attempt to solve any integrals, but do transform them (by changes of variables) into dimensionless form. Find the power α in the internal energy dependence $E = aT^{\alpha}$ on temperature (do not attempt to evaluate the proportionality constant a). What is the equivalent power for the temperature dependence of the heat capacity?

   (b) Adapt the calculation from (a) to photons, which are masless $m = 0$ and hence have energy dispersion $\epsilon(p) = pc$, where p is momentum and c is the speed of light. Find the power α in the internal energy dependence $E = aT^{\alpha}$ on temperature (do not attempt to evaluate the proportionality constant a). What is the equivalent power for the temperature dependence of the heat capacity?

Useful formulas:

- Trigonometry

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad , \quad \cosh x = \frac{e^x + e^{-x}}{2} \quad , \quad \tanh x = \frac{\sinh x}{\cosh x}$$

$$\frac{d}{dx}\sinh x = \cosh x \quad , \quad \frac{d}{dx}\cosh x = \sinh x$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\lim_{x\to 0}\sinh x = x \quad , \quad \lim_{x\to\infty}\sinh x = e^x$$

- Geometric progression:

$$|x| < 1 \quad \Rightarrow \quad \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

- Fermi-Dirac and Bose-Einstein mean occupation numbers ($\beta = 1/k_B T$):

$$f_{FD}(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)} + 1} \quad , \quad f_{BE}(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)} - 1}$$

- The density of states: summing over discrete quantum numbers $n \in \{1, 2, 3, \dots\}$ in the limit of infinite volume $V = L^3$:

$$\sum_{\mathbf{n}} \to \int_0^{\infty} dn_x \int_0^{\infty} dn_y \int_0^{\infty} dn_z = \frac{1}{8}\int_{-\infty}^{\infty} dn_x \int_{-\infty}^{\infty} dn_y \int_{-\infty}^{\infty} dn_z = \frac{1}{8}\int d^3 n$$

- Momentum quantization:

$$p = \hbar k \to \frac{\hbar\pi}{L} n$$