Qualifying exam-August 2022

Statistical Mechanics

You can use one textbook. Please write legibly and show all steps of your derivations.

Problem 1 [30 points]**:**

Consider a substance with the fundamental equation

$U = A exp\left(\alpha S + \beta M^2\right)$

where U is energy, M is the total magnetic moment, S is the entropy and A, α and β are some constants which can depend on N.

(a) [5 points] Using the 1st law of thermodynamics derive a relation for M as a function of T and H. (H is magnetic field inside the substance).
(b) [5 points] Calculate the isothermal magnetic susceptibility χ_T (T, H).
(c) [10 points] Calculate the adiabatic magnetic susceptibility χ_S (T, H). Find the small field limit $H \to 0$.
(d) [10 points] Derive a relation for Helmholtz Free energy F (T, M).

Problem 2 [20 points]:

Consider a harmonic solid with an isotropic dispersion relation given by $\omega = Bk^S$, where ω and k are the frequency and the wave number of vibrations existing in the solid, respectively. Show that the specific heat of the solid at low temperatures is proportional to $T^{3/S}$. Assume B and S are positive constants.

Problem 3 [20 points]**:**

Consider a degenerate gas (T=0K) consisting of N non-interacting relativistic electrons with a linear energy-momentum relation ($\varepsilon = v|p|$) enclosed in a volume V.

(a) (10 points) Find the Fermi energy of the gas in terms of N and V.
(b) (10 points) Find a relation relating pressure, energy, and volume of the gas.

Problem 4 [30 points]**:**

Consider a gas of non-interacting identical non-relativistic bosons with zero spin and mass *m* moving freely in a volume V.

(a) (15 points) Find the energy and heat capacity in the limit of very low temperatures where chemical potential equals zero.
Note: You do not need to evaluate all integrals and you can leave them in dimensionless form.

(b) (15 points) Explain whether Bose-Einstein condensation can occur in a two-dimensional boson gas and why. What about a one-dimensional gas?

Mathematical Formulas:

$$I_s = \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx = (s-1)!\,\zeta(s), \textit{ where } \zeta(s) \textit{ is Riemann zeta function}$$

s	1	2	3	4
$\zeta(s)$	∞	$\frac{\pi^2}{6}$	1.202	$\frac{\pi^4}{90}$