Qualifying exam-August 2020

Statistical Mechanics

You can use one textbook. Please write legibly and show all steps of your derivations.

Problem 1 [35 points]
Consider a system of N non-interacting identical non-relativistic fermions of mass m trapped in a 1D harmonic potential where the energy spectrum is given by:

$$\epsilon_n = \hbar\omega(n + \frac{1}{2})$$

The gas is in equilibrium at temperature T. For simplicity, ignore the spin degeneracy of each level.

(a) (4 points) Find the Fermi energy ε_F of the gas at T = 0 as a function of N.

(b) (8 points) Calculate the exact total energy per particle E/N at T = 0.

(c) (5 points) Calculate the grand canonical partition function $\Gamma(\mu, N)$ where μ is the chemical potential.

(d) (5 points) Calculate the grand potential. Do not attempt to evaluate the infinite sum.

(e) (5 points) Derive the average total number of particles $\bar{N}(\mu, T)$. Do not attempt to evaluate the infinite sum.

(f) (8 points) Find an explicit expression for $\bar{N}(\mu, T)$ in high temperature regime ($k_B T \ll \hbar\omega$) and $\zeta \ll 1$,

where $\zeta = e^{\beta\mu}$ is the fugacity of the gas.

You might use the following formulas for this problem:

$$\sum_{n=0}^{N-1} n = \frac{N(N-1)}{2}$$

$$\sum_{0}^{\infty} x^n = \frac{1}{1-x}$$

Problem 2 [20 points]

Assume a harmonic solid has an isotropic dispersion relation given by $\omega = Bk^S$, where ω is the frequency and k the wave number of a vibrational mode existing in a solid. Show that the specific heat of the solid at low temperatures is proportional to $T^{3/S}$. Assume B and S are positive numbers.

Problem 3 [25 points]

Consider a gas of spinless particles inside a container with volume of 1 cubic meter and in contact with a heat reservoir at temperature T. Assuming the classical Hamiltonian

$$H = \frac{p^2}{2m}$$

Where m is the particle mass, calculate:

(a) (5 points) One particle partition function Z.

(b) (20 points) The energy fluctuations per particle:

$$\overline{(E - \bar{E})^2} = \overline{E^2} - \bar{E}^2$$

Problem 4 [20 points]

A material is found to have a thermal expansivity $\alpha_P = v^{-1}(\frac{R}{P} + a/RT^2)$ and isothermal compressibility $\beta_T = v^{-1}[Tf(p) + \frac{b}{P}]$.

Where $v = V/n$ is the molar volume.

(a) (5 points) Find $f(p)$.

(b) (10 points) Find $v = v(P, T)$.

(c) (5 points) Under what condition this material is stable.

Hint: Use Maxwell relations.