Qualifying exam - August 2015

Statistical Mechanics

You can use one textbook. Please write legibly and show all steps of your derivations. Note the Formula Sheet attached.

Problem 1 [15 points]
For a classical ideal gas of identical particles, calculate the average speed $\overline{v_\perp}$ of the center of mass of a particle in the plane normal to a given axis x.

Problem 2 [15 points]
Consider a free electron gas in an imaginary *two-dimensional universe*. Suppose the gas temperature is much smaller than its Fermi temperature T_F. Present a detailed, step by step derivation of the following relations:

$$u = \frac{1}{2}\varepsilon_F, \tag{1}$$

$$p = nu, \tag{2}$$

where u is the average energy per electron, n is the number density of electrons (per unit area), p is two-dimensional pressure of the gas, and ε_F is the Fermi energy. Assume that the Pauli principle remains valid in the two-dimensional universe.

Problem 3 [28 points]
Consider a system of $N \gg 1$ localized identical particles each possessing two vibrational modes with angular frequencies ω_1 and $\omega_2 > \omega_1$. Thus the energy of each particle is

$$\varepsilon_{nm} = \frac{\hbar\omega_1}{2} + \frac{\hbar\omega_2}{2} + n\hbar\omega_1 + m\hbar\omega_2 \tag{3}$$

where the the quantum numbers n and m take the values $0, 1, 2, \ldots$. The system has been equilibrated with a thermostat at a temperature T.

1. [7 points] Calculate the partition function of the system as a function of T.
2. [7 points] Calculate the canonical average values $\bar{n}$ and $\bar{m}$.
3. [7 points] Calculate the probability of finding a given particle in a state when the first vibrational mode is frozen ($n = 0$). Sketch this probability qualitatively as a function of T and explain the physical meaning of this plot.
4. [7 points] For $\omega_2 = 2\omega_1$, calculate the probability of finding a given particle with energy larger than $4\hbar\omega_1$.

Problem 4 [32 points]

Consider a gas in equilibrium with a solid surface containing ν identical adsorption sites per unit area. The adsorption energy per site is ε regardless of whether neighboring adsorption sites are occupied or vacant. The temperature of the system is T and the chemical potential of particles in the gas is μ.

1. [8 points] Apply the grand canonical formalism to calculate the average number of adsorbed particles n per unit area as a function of T and μ.

2. [8 points] Calculate the root-mean-square fluctuation of n:

$$\Delta n \equiv \left(\overline{(n-\bar{n})^2}\right)^{1/2}. \tag{4}$$

3. [8 points] Calculate the relative fluctuation

$$v \equiv \frac{\Delta n}{\bar{n}}. \tag{5}$$

4. [8 points] Qualitatively sketch $\bar{n}$ and v as functions of μ at a constant temperature and explain the physical meaning of these plots.

Formula Sheet

Moments of the Gaussian function:

$$M_n = \int_0^\infty x^n e^{-x^2} dx. \tag{6}$$

Selected values: $M_0 = \sqrt{\pi}/2$, $M_1 = 1/2$, $M_2 = \sqrt{\pi}/4$, $M_3 = 1/2$, $M_4 = 3\sqrt{\pi}/8$, $M_5 = 1$, $M_6 = 15\sqrt{\pi}/16$.