Qualifying Exam, Quantum Mechanics, Jan 2026

1. (40 points) Consider a ring of radius R with a negligible cross section. It lies within the xy plane and is centered at the origin. A particle of mass μ slides freely on the ring. The Hamiltonian is

$$H = \frac{L_z^2}{2\mu R^2}$$

Here $L_z = -i\hbar\frac{\partial}{\partial\phi}$ is the angular momentum along z, $\phi \in [0, 2\pi)$ is the polar angle. Let $|\phi'\rangle$ be the eigen ket describing the particle localized at angle ϕ'. Hint: use your knowledge of angular momentum.

(a) What are the energy eigen values? What is the degeneracy of the first excited state(s)?

At time $t = 0$, the particle is prepared in a state $|\psi\rangle$ with wave function $\langle\phi'|\psi\rangle = N[1 - e^{-i\phi'} + 2e^{i3\phi'}]$, where N is a real, positive constant.

(b) Fix N so that $\langle\psi|\psi\rangle = 1$. What is the probability of finding the particle with $0 < \phi < \pi$?

(c) Compute the variance of L_z, $\langle(\Delta L_z)^2\rangle = \langle L_z^2\rangle - \langle L_z\rangle^2$, in state $|\psi\rangle$.

(d) Find the wave function at later time $t > 0$. You may use shorthand notation $\omega = \hbar/2\mu R^2$.

2. (30 points) A hydrogen atom is placed in a weak, uniform, and static electric field $\mathscr{E}\hat{z}$ with $\mathscr{E} > 0$. Use the standard notation $|nlm\rangle$, neglect the electron spin and all relativistic corrections. The effect of the electric field on the $n = 2$ energy level can be described by the following effective Hamiltonian

$$H = -\frac{R_y}{4}\big[|200\rangle\langle 200| + |210\rangle\langle 210|\big] + 3e\mathscr{E}a\big[|200\rangle\langle 210| + |210\rangle\langle 200|\big]$$

where $R_y = 13.6\text{eV}$, $e < 0$ is the electron charge, and a is the Bohr radius. States $|21 \pm 1\rangle$ are not affected by $\mathscr{E}$ to the leading order approximation.

(e) What is the ground state of H?

(f) What happens to the $n = 2$ energy level? Sketch to show the splitting, if any, and degeneracy etc.

(g) In the ground state of H, at which point along the z-axis (other than the origin) is the probability density of finding the electron zero? Hint: $\psi_{nlm} = R_{nl}(r)Y_l^m(\theta, \phi)$. Along the z-axis, $\cos\theta = 0$ or π.

3. (30 points) A beam of neutrons (charge neutral, spin 1/2 particles) are prepared in an unknown, pure, spin state $|\psi\rangle$. Three students, Alice, Bob, and Charlie, are drafted to figure out what $|\psi\rangle$ is. Alice performs SG_z experiments to measure S_z and finds the probability of $S_z = +\hbar/2$ is $1/2$. Similarly, Bob measures S_x and finds the probability of $S_x = +\hbar/2$ is $3/4$. Finally, Charlie measures S_y and finds the probability of $S_y = +\hbar/2$ is $(1/2 - \sqrt{3}/4)$. These measurements are not sequential. In other words, the students carry out independent measurements on different neutrons prepared in the same state.

(h) Use the data to determine $|\psi\rangle$. Hint: you may parameterize $|\psi\rangle = \cos\frac{\theta}{2}|+\rangle + e^{i\phi}\sin\frac{\theta}{2}|-\rangle$, with $\theta \in [0, \pi]$. It is helpful to visualize the state using the Bloch sphere.

Taped on the rusty magnet in their lab was a handwritten note,

$$e^{-i\pi S_y/\hbar} S_z e^{i\pi S_y/\hbar} = -S_z.$$

(i) Help the students decide if the note is a legit identity or some crude joke. Justify your answer.