Qualifying Exam, Quantum Mechanics, Jan 2025

1. (15 points) A beam of spin-1/2 atoms pass through a sequence of thirteen Stern-Gerlach apparatus labelled SG_0 to SG_{12}. SG_n has magnetic field aligned in the xz-plane, forming an angle $\theta_n = n\pi/12$ from the z axis with $n = 0, 1, ..., 12$. SG_n makes a selective measurement and only allows the spin state

$$|S_n; +\rangle = \cos\frac{\theta_n}{2}|+\rangle + \sin\frac{\theta_n}{2}|-\rangle$$

to go through. Here the shorthand notation $|\pm\rangle = |S_z; \pm\rangle$. What is the probability that the spin-1/2 atoms, initially in the state $|+\rangle$, will pass all 13 SG apparatus?

2. (15 points) The electron wave function in a hydrogen atom has the form

$$\psi(\mathbf{x}) = C(i\psi_{100}(\mathbf{x}) - \sqrt{3}\psi_{21-1}(\mathbf{x}))$$

where $\psi_{nlm}(\mathbf{x}) = \langle\mathbf{x}|nlm\rangle = R_{nl}(r)Y_l^m(\theta, \phi)$ are the (properly normalized) energy eigenstates in standard notation, and C is a constant to be determined. What is the average energy of the electron in this state?

3. (30 points) A diatomic molecule consists of two identical atoms. Assume the chemical bond is so stiff that to a very good approximation it is a rigid rotor with fixed bond length. The Hamiltonian then is $H = \boldsymbol{L}^2/2I$, where $\boldsymbol{L}$ is the angular momentum operator and I is the moment of inertia.

(a) What are the energy eigenvalues, and what is the degeneracy of each energy level?

(b) A microwave pulse excites the molecules into the rotational state $|l = 1, m = -1\rangle$. Here $|l, m\rangle$ is the simultaneous eigenket of $\boldsymbol{L}^2$ and L_z with eigenvalues $l(l+1)\hbar^2$ and $m\hbar$, respectively. When L_x, the angular momentum along x, is measured, what is the variance $\langle(\Delta L_x)^2\rangle = \langle L_x^2\rangle - \langle L_x\rangle^2$?

4. (40 points) A one-dimensional simple harmonic oscillator with mass m and angular frequency ω has eigenenergies $E_n = (n + 1/2)\hbar\omega$, and the corresponding eigenkets are $|n\rangle$, $n = 0, 1, 2....$ You can directly quote results on the position $\hat{x}$, momentum $\hat{p}$, raising and lowering operators $\hat{a}^\dagger$, $\hat{a}$, the ground state wave function (e.g. Sakurai 2.151) and the time evolution of simple harmonic oscillator from a textbook.

The oscillator is initially in the ground state $|0\rangle$. Now we give it an instantaneous **kick** which imparts momentum $\hbar k$ (a real constant) to the system. To describe this, introduce a "boost" operator $\hat{b}_k = e^{ik\hat{x}}$. After the kick, we set the time $t = 0$, so the state $|\psi(t = 0)\rangle = \hat{b}_k|0\rangle = e^{ik\hat{x}}|0\rangle$. View it as a superposition, $|\psi(t = 0)\rangle = \sum_n c_n|n\rangle$, we can find $c_n = \langle n|\hat{b}_k|0\rangle$ as follows.

(I) Evaluate $c_0 = \langle 0|\hat{b}_k|0\rangle$. You may choose a convenient basis.

(II) Show that the commutator

$$[\hat{a}, \hat{b}_k] = ik\sqrt{\frac{\hbar}{2m\omega}}\hat{b}_k.$$

Then, by examining the matrix element $\langle n - 1|[\hat{a}, \hat{b}_k]|0\rangle$ show that we have recursion relation

$$c_n = ik\sqrt{\frac{\hbar}{2m\omega}}\frac{1}{\sqrt{n}}c_{n-1}.$$

Finally, iterate and use the recursion relation to find c_n in terms of c_0 you computed in (I).

(III) If we make an energy measurement right after the kick ($t = 0$), what is the probability P_n of finding the value E_n?

(IV) For $t > 0$, the system continues to evolve according to the simple harmonic oscillator Hamiltonian. Show the expectation value of the position at time t is given by

$$x(t > 0) = \frac{\hbar k}{m\omega}\sin(\omega t).$$