Quantum Mechanics Qualifying Exam

Spring 2024

January 10 (9:00 am - 12:00 pm)

1. Consider a physical system with a three-dimensional Hilbert space. The Hamiltonian is represented by the matrix:

$$H = \epsilon \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

where ϵ is an energy scale.

(a) What are the possible outcomes of energy measurements?

(b) The system is prepared in the state represented by the vector:

$$\psi \propto \begin{pmatrix} i \\ 1 \\ -i \end{pmatrix}$$

What is the expectation value $\langle H \rangle$ of energy in this state? What is the standard deviation of energy measurements in this state?

2. Three identical particles of mass m are placed in a two-dimensional symmetric harmonic potential with harmonic frequency ω. Find the three lowest energy states of these three particles and their degeneracy if:

(a) the particles are spinless bosons,

(b) the particles are spinless fermions.

3. Two localized and non-interacting electrons are placed in a non-uniform external magnetic field which points everywhere in the z-direction. Their spin dynamics is governed by the Hamiltonian

$$H = \epsilon_1 \sigma_1^z + \epsilon_2 \sigma_2^z$$

where ϵ_i are energy scales and $\sigma_i^{x,y,z}$ are the Pauli matrices associated with the electrons $i = 1, 2$. The electrons are initially entangled in a spin singlet state:

$$|\psi_0\rangle = \frac{1}{\sqrt{2}} \Big(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \Big)$$

(a) What is the probability that the two electrons will be detected in the spin singlet state at a later time t?

(b) Construct the state vectors of all spin-triplet eigenstates of the total spin along the field direction. What are the probabilities of detecting the two electrons in each triplet state at any time t?

(c) What are the possible outcomes of measuring the total spin in a direction perpendicular to the field, say $S^x = S_1^x + S_2^x$? How do the probabilities of these outcomes depend on time?

(d) How do these results change if we reorient the magnetic field in a different direction, y for example?

4. A particle is trapped in a certain one-dimensional stationary bound state. The statistics of the particle's position x was measured by an array of detectors, so we know the (normalized) position probability $P(x)$ per unit length:

$$P(x) = Ax^2 e^{-bx^2}$$

(a) Show that the wavefunction must be real (up to a global constant phase $e^{i\theta}$). [Hint: Construct the probability current density $j(x)$ for an arbitrary state $\psi(x)$ in one dimension. What must $j(x)$ be in a stationary state? What property must $\psi(x)$ have in order to produce such a $j(x)$?]

(b) What is the wavefunction of the particle with the given $P(x)$? Determine the normalization constant A in terms of b.

(c) If the particle has mass m, what potential $V(x)$ produces this stationary state?

(d) Find the normalized wavefunction of the stationary state in the next energy level (with higher energy).

5. A particle of mass m is inside a spherical potential well of radius a and depth V_0. The potential energy as a function of radius r is:

$$V(r) = \begin{cases} -V_0 & , \quad r < a \\ 0 & , \quad r > a \end{cases}$$

It is known that the angular momentum of the particle is zero.

(a) Write the Schrodinger equation for the full wavefunction $\Psi(\mathbf{r})$ in the spherical coordinate system.

(b) The potential energy is spherically symmetric, so the stationary wavefunction can be constructed as a product $\Psi(\mathbf{r}) = \psi(r) Y_l^m(\theta, \phi)$ of the radial and spherical harmonic factors. Derive the Schrodinger equation for the radial part $\psi(r)$ knowing that the angular momentum is zero.

(c) Construct the radial equation for the modified wavefunction $R(r) = r\psi(r)$. Then, proceed to solve this equation in the regions $r < a$ and $r > a$ for bound states, $-V_0 < E < 0$. Obtain the boundary conditions and solve for all constants in terms of one (the last constant can be determined only by normalization, which you should not attempt).

(d) Derive the equation from which one would numerically find the energy spectrum of bound states with zero angular momentum. Proceed by requiring $R(0) = 0$, with the help of the results in part (c). Ideally, show that:

$$ka = n\pi - \arcsin\left(\frac{\hbar k}{\sqrt{2mV_0}}\right)$$

where

$$k = \sqrt{\frac{2m(E + V_0)}{\hbar^2}}$$

Useful formulas:

- Gaussian integral:

$$\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}$$

- Harmonic oscillator:

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$$

$$a = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega x + ip) \quad , \quad a^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega x - ip)$$

- Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$e^{i\boldsymbol{\sigma}\hat{\mathbf{n}}\theta} = \cos\theta + i(\boldsymbol{\sigma}\hat{\mathbf{n}})\sin\theta$$

- Laplacian in the spherical coordinate system:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

Trigonometry:

$$\arctan(x) = \arcsin\left(\frac{x}{\sqrt{1+x^2}}\right)$$