Qualifying Exam, Quantum Mechanics, January 2023

Notations follow Sakurai and Napolitano, 3rd Ed.

1. (30 points) A one-dimensional harmonic oscillator is described by the Hamiltonian

$$H = \frac{p_x^2}{2m} + \frac{1}{2}m\omega^2 x^2,$$

where p_x is the momentum along x. The eigenenergies are $E_n = (n + 1/2)\hbar\omega$ and the eigenkets are $|n\rangle$, with $n = 0, 1, 2....$ Suppose the system is in a pure state $|\psi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$.

(a) Compute the expectation value of the position, $\langle x\rangle$.

(b) What is the average kinetic energy $\langle p_x^2/2m\rangle$ of the oscillator?

(c) Suppose the system is instead in a mixed state with density operator $\rho = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$. Compute the ensemble average of H.

2. (30 points) The electron in an excited hydrogen atom is described by the wave function

$$\psi(x, y, z) = \frac{1}{8\sqrt{\pi a^5}}(ix + z)e^{-\frac{r}{2a}},$$

where $r = \sqrt{x^2 + y^2 + z^2}$ is the distance from the nucleus (proton), and a is a given constant. ψ is properly normalized. For this problem, you do not need to worry about the spin of the electron.

(a) If the radial position r is measured, what is the most probable value?

(b) If the orbital angular momentum L_z is measured, what are the possible outcomes and the corresponding probabilities?

(c) Find $\langle L_x\rangle$ in state $\psi(x, y, z)$, where L_x is the x-component of the orbital angular momentum.

3. (20 points) A beam of neutrons (which are spin 1/2 particles), prepared in the spin-up state $|S_z; +\rangle$ and having the same velocity, pass through a region of uniform magnetic field $B\hat{x}$. The Hamiltonian is $H = -\gamma_n S_x B$ where γ_n is the gyromagnetic ratio of the neutron, and $S_x = \frac{\hbar}{2}\sigma_x$ is the spin along $\hat{x}$. Let T be the time for the neutron to transverse the $B\hat{x}$ region.

(a) Find the spin state of the neutrons after they leave the magnetic field region.

(b) Down stream, a Stern-Gerlach apparatus measures the spin S_y of the neutrons coming out of the $B\hat{x}$ region. What is the probability of finding the value $\hbar/2$?

4. (20 points) There are three kinds of neutrinos, the electron neutrino ν_e, the muon neutrino ν_μ, and the tau neutrino ν_τ. In a toy model, they are conjectured to be orthogonal quantum states of a single system, i.e., different superpositions of three orthonormal energy eigenstates $|\phi_i\rangle$ with eigenenergy E_i, $i = 1, 2, 3$. For example, the electron- and muon-neutrino states are

$$|\nu_e\rangle = \frac{1}{2}|\phi_1\rangle + \frac{\sqrt{3}}{4}|\phi_2\rangle, \quad |\nu_\mu\rangle = \frac{3}{4}|\phi_1\rangle - \frac{\sqrt{3}}{4}|\phi_2\rangle - \frac{1}{2}|\phi_3\rangle.$$

Assume non-relativistic quantum mechanics applies, E_i are given constants, and $E_3 > E_2 > E_1$.

Suppose at time $t = 0$, the system is in state $|\nu_e\rangle$. According to this toy model,

(a) What is the probability of finding the system in state $|\nu_\mu\rangle$ at later time $t > 0$?

(b) Is there a time instant at $t > 0$ when there is 100% chance to observe the electron neutrino? Explain.

$$|\nu_e\rangle = \frac{1}{2}|\phi_1\rangle + \frac{\sqrt{3}}{2}|\phi_2\rangle$$

IS PROPERLY NORMALIZED