Quantum Mechanics Qualifying Exam, January 2020

1. [20 pts] Consider a proton described by the Hamiltonian H

$$H = \epsilon \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$$

in the standard representation that diagonalizes the z-projection of spin (S_z). ϵ is a constant with units of energy.

(a) Suppose we measure the proton's energy at time $t = 0$. What possible values could we obtain? If we measure the energy of this proton again at time $t' > 0$, how will it be related to the energy measured at $t = 0$?

(b) Suppose the proton is equaly likely to be found in its two possible stationary states at $t = 0$. What is the most general spinor that represents such a state? What does this spinor become at $t = 10\,\hbar/\epsilon$?

(c) Suppose that a measurement of S_x at time $t = 0$ yields $\hbar/2$. What is the probability of finding $\hbar/2$ in a subsequent measurement of S_z? What would be the probability of finding $-\hbar/2$ in a measurement of S_y? What would be the probability of finding $-\hbar/2$ in a measurement of S_x?

2. [20 pts] At $t = 0$, a particle of mass m in a one-dimensional world is in a stationary state given by the wavefunction $\psi(x) = Ae^{-(2\xi+3)^2}$, where $\xi = x/a$ is the "dimensionless" position measured in the units of a. Determine the normalization constant A. Then:

(a) Calculate the expectation values $\langle x \rangle$, $\langle p \rangle$, $\langle x^2 \rangle$, $\langle p^2 \rangle$, and the product $\Delta x \Delta p$ of position and momentum uncertainties.

(b) Write down the Hamiltonian H of the particle whose ground state is given by ψ. What is the ground-state energy of the particle with this Hamiltonian?

(c) Where is the particle most likely to be found?

(d) What is $\langle p \rangle$ if the wave function of the particle is $\psi(x) = A'e^{-2\xi^2}/\xi^4$ instead?

3. [10 pts] The wave function of a particle moving in three dimensions is $\psi(r, \theta, \phi) = Ae^{-r/a}$, where a is a constant length. Calculate the radius r where the particle is most likely to be found. What is the angular momentum of the particle?

4. [10 pts] Obtain the eigenfunctions and eigenvalues of a two-dimensional quantum mechanical rigid rotor of moment of inertia I. Does the system have zero point energy? Explain your answer.

5. [20 pts] An electron is moving in the "intrinsic" potential $V(\mathbf{r}) = \kappa(2x^2 + 5y^2 + z^2)$ and additionally subjected to a constant electric field $\mathbf{E} = \alpha(3\hat{\mathbf{x}} + \hat{\mathbf{y}} + 9\hat{\mathbf{z}})$. What are the electron's ground state energy and wave function? Write down the Hamiltonian which includes $V(\mathbf{r})$, the electric field, and also a constant magnetic field $\mathbf{B} = \beta \cdot 5\hat{\mathbf{z}}$. What is the ground state energy of a neutron in the same conditions?

6. [20 pts] Consider a particle in the state given by the wave function $\psi(x, y, z) = Ae^{-2r^2/a^2}(x^2 + y^2 + z^2 + xy)$, where A is the normalization constant. What are the possible angular momentum quantum numbers (l, m) that the particle can have in this state? Calculate the angular momentum expectation value $\langle L_z \rangle$.