Quantum Mechanics Qualifying Exam

Spring 2018

January 18 (9:00 am - 12:00 pm), Exploratory Hall 1004

1. Evaluate the x-p uncertainty product $\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle$ for a one-dimensional particle confined between two rigid walls, experiencing the potential:

$$V(x) = \left\{ \begin{array}{ccc} 0 & , & 0 < x < a \\ \infty & , & \text{otherwise} \end{array} \right\}$$

   Do this for the ground and all excited states.

2. A box containing a particle is divided into a right and left compartment by a thin partition. If the particle is known to be on the right (left) side with certainty, its state is represented by a "position" eigenstate $|R\rangle$ ($|L\rangle$), where we have neglected spatial variations within each half of the box. The most general state vector can be written as:

$$|\alpha\rangle = |R\rangle\langle R|\alpha\rangle + |L\rangle\langle L|\alpha\rangle = C_R|R\rangle + C_L|L\rangle$$

   where $C_R = \langle R|\alpha\rangle$ and $C_L = \langle L|\alpha\rangle$ can be regarded as "wavefunctions". The particle can tunnel through the partition. This tunneling effect is characterized by the Hamiltonian:

$$H = \Delta\Big(|L\rangle\langle R| + |R\rangle\langle L|\Big)$$

   where Δ is a real number with units of energy.

   (a) Find the normalized vectors of stationary states. What are the corresponding energies?

   (b) Suppose the system is in some arbitrary initial state $|\alpha\rangle$ at $t = 0$, specified by $C_R = \langle R|\alpha\rangle$ and $C_L = \langle L|\alpha\rangle$ as above. What is the system's state at any time $t > 0$?

   (c) Suppose that the particle is on the right side of the partition with certainty at $t = 0$. What is the probability of observing the particle on the left side as a function of time?

3. Calculate the correlation function

$$C(t) = \langle \hat{x}(t)\hat{x}(0)\rangle$$

   in the ground state of a one-dimensional linear harmonic oscillator (of mass m and harmonic frequency ω), where $\hat{x}(t)$ is the position operator in the Heisenberg picture.

4. Calculate the three lowest energy levels together with their degeneracies for the following systems (the particles have mass m):

   (a) Three non-interacting spin $\frac{1}{2}$ identical fermions in a 1D box of length L.

   (b) Three non-interacting spin $\frac{1}{2}$ distinguishable particles in a 1D box of length L.

   (c) Four non-interacting spin $\frac{1}{2}$ identical fermions in a 1D box of length L.

   (d) Four non-interacting spin $\frac{1}{2}$ distinguishable particles in a 1D box of length L.

5. Consider an orbital angular momentum eigenstate $|l = 2, m = 0\rangle$. Suppose this state is rotated by an angle β about the y-axis. Find the probability for the new state to be found with $m = 0, \pm 1, \pm 2$.

   Hint: This is a challenging problem. Follow these steps for partial credit:

   (a) Let $\mathcal{R}_{\beta,\hat{\mathbf{y}}}$ be the operator that rotates by β about the y-axis. Expand $\mathcal{R}_{\beta,\hat{\mathbf{y}}}$ in the basis of the L_y eigenstates $|2, m_y\rangle$, where $m_y \in \{-2, -1, 0, 1, 2\}$. Just write the expansion using Dirac notation (you don't know what the states $|2, m_y\rangle$ look like yet).

   (b) Construct the 5-dimensional ($2l + 1 = 5$ for $l = 2$) matrix representation of the operators $L_+ = L_x + iL_y$ and $L_- = L_x - iL_y$ in the usual basis that diagonalizes L_z (use the known effect of $L_\pm$ on the normalized eigenstates of L_z). Then, construct the matrix representation of L_y.

   (c) Diagonalize L_y, i.e. find the normalized eigenvectors of L_y (representation of all $|2, m_y\rangle$) from the matrix constructed in part (b). Since the eigenvalues $\hbar m_y$ of L_y are known, you can proceed straight to the formulation of 5 equations with 5 unknowns for the 5 components of an eigenvector corresponding to a generic m_y. This system is not hard to solve: set one eigenvector component to 1 and determine the others. Then, substitute the 5 possible values for m_y, write the 5 eigenvectors and normalize each one of them.

   (d) Use (a) and (c) to rotate the initial state $|2, 0\rangle \equiv |0, m_z = 0\rangle$ (note m_z here - not m_y) and represent it in the basis of $|2, m\rangle \equiv |2, m_z\rangle$. Calculate the probabilities P_m of measuring $m \equiv m_z$ in the rotated state.