QM Qualifier Jan. 2016 name______________________________

You are allowed to bring one textbook of your choice.

Problem 1 (25 points) A spin-1/2 particle is in the eigenstate $|+\rangle$ corresponding to the eigenvalue ħ/2 of the of S_z operator. A magnetic field in the *x*-*z* plane is turned on at time $t = 0$; it makes an angle θ with the *z*-axis and has Larmor frequency ω.

(a) Find the state vector at any given time *t*.

(b) Calculate the expectation value of spin $\langle \vec{S}(t) \rangle$ at any time *t*.

Problem 2 (25 points). The operator $D(\hat{n},\varphi) = e^{-i\frac{\varphi}{2}\vec{\sigma}\cdot\hat{n}}$ represents a rotation by angle φ about the unit vector $\hat{n}$. Make two consecutive rotations, $D(\hat{n}_1,\varphi_1)$ followed by $D(\hat{n}_2,\varphi_2)$. Set the rotation axes in the *x*-*y* plane as, $\hat{n}_1 = (1,0,0)$, $\hat{n}_2 = (-\cos\theta, -\sin\theta, 0)$, and the rotation angles to $\varphi_1 = \varphi_2 = \pi$.

(a) Prove that $U = D(\hat{n}_2,\varphi_2)D(\hat{n}_1,\varphi_1)$ is a rotation operator. Find the rotation axis $\hat{n}$ and angle φ.

(b) Rotate both axes $\hat{n}_1$ and $\hat{n}_2$ about the *z*-axis by an angle α to $\hat{n}_1'$ and $\hat{n}_2'$. Find the rotation axis and angle for the operator $U' = D(\hat{n}_2',\pi)D(\hat{n}_1',\pi)$.

(c) If both rotation angles in (a) deviate from π by a small amount 2ε, the operator becomes $U'' = D(\hat{n}_2,\pi+2\varepsilon)D(\hat{n}_1,\pi+2\varepsilon)$. Calculate the trace $tr(U^+U'')$. Make sure to obtain $\lim_{\varepsilon\to 0} tr(U^+U'') = tr(U^+U)$.

Problem 3 (25 points) A coherent state of the one-dimensional simple harmonic oscillator is defined as an eigenstate of the annihilation operator a: $a|\lambda\rangle = \lambda e^{i\varphi}|\lambda\rangle$, where λ and φ are real.

(a) Prove that

$$|\lambda\rangle = \sum_{n=0}^{\infty} \frac{\lambda^n e^{in\varphi}}{\sqrt{n!}} e^{-\lambda^2/2} |n\rangle,$$

where $|n\rangle$ denotes a number state.

(b) Prove that

$$\langle x|\lambda\rangle = \frac{1}{\sqrt[4]{\pi}\sqrt{x_0}} e^{-\frac{(x-x_c)^2}{2x_0^2} + ik_c x}$$

is a wavefunction that satisfies the eigen-equation of the annihilation operator represented in position space, where

$$x_c = \sqrt{2}x_0\lambda\cos\varphi,\; k_c = \sqrt{2}\frac{\lambda}{x_0}\sin\varphi,\text{ and } x_0 = \sqrt{\frac{\hbar}{m\omega}}$$

Problem 4 (25 points) A system consists of two identical spin 1/2 fermions at rest. Let $\vec{S}_1$ and $\vec{S}_2$ be the individual particle spin operators. The spin-spin coupling Hamiltonian is $H = \gamma\vec{S}_1 \cdot \vec{S}_2$, where γ is a real constant.

(a) Find the eigenstates and eigenvalues of H.

(b) If one measures S_{1z} in the ground state, what are the possible measured values and their corresponding probabilities?

(c) Pick one of the possible states after (b) and measure S_{2z} on it. What are the possible measurement outcomes and their corresponding probabilities?