Ph.D. Qualifier Exam

Quantum Mechanics, Spring 2015

Jan 15 (9 am - noon)

1. (25 points) A static magnetic field in the x-direction is applied to a spin-1/2 particle; ω is the corresponding Larmor frequency. Discuss the following properties of observables that are independent of the initial state.

   (a) Write three observables of the particle whose expectation values are time-invariant (and explain).

   (b) Write two observables whose expectation values are periodic functions of time with period of $2\pi/\omega$ (and explain).

2. (25 points) In a two-state system, the operator $U(\varphi, \hat{\mathbf{n}}) = \exp\left(-i\frac{\varphi}{2}\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}\right)$ imparts a rotation on state vector, where φ denotes the rotation angle and the unit vector $\hat{\mathbf{n}}$ defines the rotation axis.

   (a) Prove that

   $$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

   represents a rotation operator. Find the rotation axis and angle.

   (b) Prove that the operator in (a) is the transformation operator that changes the spin S_z-eigenbasis to S_y-eigenbasis.

   (c) Find the matrix form of $[A, B]$ for operators A and B. A's matrix form

   $$A = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

   is given in the S_z-eigenbasis and

   $$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

   is in the S_y-eigenbasis.

3. (25 points) One dimensional simple harmonic oscillator is in a coherence state $|\lambda\rangle$, which is the eigenket of annihilation operator, $a|\lambda\rangle = \lambda|\lambda\rangle$. Use ω to denote the oscillator's angular frequency.

(a) Calculate the expectation value of the oscillator's total energy.

(b) Calculate the uncertainty of the total energy.

(c) Let $\lambda = \sqrt{n}$, where n is a non-zero integer. Compare the coherent state $|\lambda\rangle$ and the number operator eigenstate $N|n\rangle = n|n\rangle$ in the list below:

	$\|\lambda\rangle$	$\|n\rangle$
energy expectation value		
energy uncertainty value		
stationary (yes or no)		
minimum uncertainty state (yes or no)		

4. (25 points) An ensemble of spin-1/2 particles is in an unknown state. Answer questions according to the expectation value of spin angular momentum given for 2 cases in (a) and (b) below, where the Cartesian coordinates are defined by unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$. Is the ensemble in a pure or mixed state? Find the state ket if it is a pure state, and density matrix if mixed state.

(a)

$$\langle \mathbf{S} \rangle = \frac{\hbar}{4}(\hat{\mathbf{j}} + \hat{\mathbf{k}})$$

(b)

$$\langle \mathbf{S} \rangle = \frac{\hbar}{2\sqrt{2}}(\hat{\mathbf{i}} + \hat{\mathbf{j}})$$