Qualifying Exam, Quantum Mechanics, Aug 2026

1. (40 points) A spin qubit is described by the following Hamiltonian,

$$H = -B\left[\frac{1}{\hbar}S_z^2 + S_z + S_x\right]$$

where B is a real positive constant, $S_{x,y,z}$ are standard spin-$\frac{1}{2}$ operators.

(a) Find the eigen energies of the spin qubit.

(b) In the ground state, what is the probability of finding $S_z = +\hbar/2$?

Suppose at time $t = 0$, the qubit is prepared in state $|S_z; +\rangle$.

(c) What is the probability of finding $S_z = -\hbar/2$ at later time $t > 0$?

(d) What is the expectation value of S_y at time $t > 0$?

2. (30 points) A particle of mass m is confined by a two-dimensional, anisotropic trap potential. Its Hamiltonian is given by

$$H = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2}m\omega^2(x^2 + 4y^2)$$

where ω is a positive real constant. You can quote standard results for the simple harmonic oscillator.

(a) Express H in terms of the lowering (i.e. annihilation) operators a_x, a_y and their adjoints $a_x^\dagger, a_y^\dagger$. Discuss the degeneracy of the lowest three energy levels.

(b) In the Heisenberg picture, find operator $a_y(t)$ and show it oscillates with frequency 2ω.

(c) The orbital angular momentum operator is defined as $L_z = xp_y - yp_x$. Express L_z in terms of the creation and annihilation operators $a_x, a_y, a_x^\dagger, a_y^\dagger$.

3. (30 points) The electron inside a hydrogen atom is described by the following wave function

$$\psi = R_{21}(r)\left[\sqrt{\frac{2}{5}}Y_1^0(\theta,\phi)\chi_+ + \sqrt{\frac{3}{5}}Y_1^1(\theta,\phi)\chi_-\right]$$

Here $R_{n\ell}(r)$ are the standard radial wave functions, $Y_\ell^m(\theta,\phi)$ are spherical harmonics, and $\chi_\pm$ are the standard Pauli spinors. You can directly quote their expressions from a textbook. Let $\mathbf{J} = \mathbf{L} + \mathbf{S}$ be the total angular momentum as the sum of the orbital angular momentum $\mathbf{L}$ and the spin $\mathbf{S}$.

(a) What is the most probable radius r to find the electron (e.g., in units of the Bohr radius)?

(b) Show that $\langle L_x\rangle = \langle L_y\rangle = 0$ in this state.

(c) If $\mathbf{J}^2$ is measured, what are the possible values and their respective probabilities?

The Clebsch-Gordan coefficients below may be useful. On the left is $|j, m\rangle$, on the right are $|m_\ell\rangle|m_s\rangle$.

$$\left|\frac{3}{2}, +\frac{3}{2}\right\rangle = |1\rangle\left|+\frac{1}{2}\right\rangle, \quad \left|\frac{3}{2}, -\frac{3}{2}\right\rangle = |-1\rangle\left|-\frac{1}{2}\right\rangle$$

$$\left|\frac{3}{2}, +\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}|0\rangle\left|+\frac{1}{2}\right\rangle + \sqrt{\frac{1}{3}}|1\rangle\left|-\frac{1}{2}\right\rangle$$

$$\left|\frac{3}{2}, -\frac{1}{2}\right\rangle = \sqrt{\frac{1}{3}}|-1\rangle\left|+\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}|0\rangle\left|-\frac{1}{2}\right\rangle$$

$$\left|\frac{1}{2}, +\frac{1}{2}\right\rangle = \sqrt{\frac{1}{3}}|0\rangle\left|+\frac{1}{2}\right\rangle - \sqrt{\frac{2}{3}}|1\rangle\left|-\frac{1}{2}\right\rangle$$

$$\left|\frac{1}{2}, -\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}|-1\rangle\left|+\frac{1}{2}\right\rangle - \sqrt{\frac{1}{3}}|0\rangle\left|-\frac{1}{2}\right\rangle$$