Qualifying Exam, Quantum Mechanics, Aug 2025

1. (30 points) A free particle of mass m is confined within the one-dimensional region $0 \leq x \leq L$ (i.e., inside an infinite well). At time $t = 0$, the wave function has the form

$$\psi(x, t = 0) = A \left[1 + \cos(\frac{\pi x}{L})\right] \sin(\frac{\pi x}{L})$$

where A is a real, positive constant.

(a) What is the value of A in order for $\psi(x, t = 0)$ to be normalized?

(b) If the kinetic energy of the particle is measured at $t = 0$, what are the possible values and what is its expectation value?

(c) Find $\psi(x, t)$, the wave function at later time $t > 0$. Determine the "quantum revival time," the time it takes for $|\psi(x, t)|^2$ to return to $|\psi(x, 0)|^2$.

2. (30 points) A hydrogen atom is in a $2P_{1/2}$ state where the principal quantum number $n = 2$, the orbital angular momentum quantum number $\ell = 1$, and the total angular momentum ($\mathbf{J} = \mathbf{L} + \mathbf{S}$) quantum number $j = 1/2$, $m = 1/2$. According to the theory of angular momentum,

$$|j = 1/2, m = 1/2\rangle = \sqrt{2/3}|m_\ell = 1, m_s = -1/2\rangle - \sqrt{1/3}|m_\ell = 0, m_s = 1/2\rangle.$$

The notation here follows Sakurai 3.8.3. Feel free to quote Sakurai Appendix B.5, B.6 etc. Neglect all relativistic corrections (including the spin-orbit coupling) to the energy and wave functions.

(d) Evaluate $\langle J_x \rangle$, the expectation value of J_x (the x component of $\mathbf{J}$) in this state.

(e) What is the most probable radius to find the electron? Express your result in Bohr radius a_0.

(f) Show that the probability per solid angle of finding the electron at any spherical direction (θ, ϕ) (regardless of the radial position r and spin m_s) is a constant, $1/4\pi$. Hint: express the state as superpositions of $|n, \ell, m_\ell\rangle \otimes |m_s\rangle$ and recall $\langle \mathbf{x}'|n, \ell, m_\ell\rangle = R_{n\ell}(r) Y_\ell^m(\theta, \phi)$.

3. (40 points) Inside Google's Willow there are arrays of transmon qubits. A transmon is essentially a quantum circuit consisting of a Josephson junction, a non-linear inductor with characteristic inductance L, connected to a capacitor with capacitance C. It can be described by the following Hamiltonian,

$$\hat{H} = \hbar\omega \left(\frac{\lambda}{2} \hat{n}^2 - \frac{1}{\lambda} \cos \hat{\phi} \right).$$

Here $\hat{n}$ is the number operator (the number of Copper pairs) and $\hat{\phi}$ is the flux operator (the magnetic flux in units of $\phi_0 = \hbar/2e$). These two dimensionless Hermitian operators obey the commutation relation

$$[\hat{\phi}, \hat{n}] = i$$

The constant $\omega = \sqrt{1/LC}$ is the oscillation frequency, and the dimensionless parameter λ is the impedance $\sqrt{L/C}$ measured in units of $\hbar/(2e)^2 = 1.027 k\Omega$. You do **not** need to understand the device or the E&M details to solve this problem.

First, let us approximate $\hat{H}$ by Taylor expansion of the cosine term to the quadratic order,

$$\hat{H} \simeq \hat{H}_2 = \hbar\omega \left(\frac{\lambda}{2} \hat{n}^2 + \frac{1}{2\lambda} \hat{\phi}^2 - \frac{1}{\lambda} \right).$$

Now you should recognize the analogy between $\hat{H}_2$ and the simple harmonic oscillator via the correspondence $\hat{\phi} \leftrightarrow \hat{x}$ and $\hat{n} \leftrightarrow \hat{p}$. The analogy motivates us to define the annihilation/ladder operator

$$\hat{a} = \sqrt{\frac{1}{2\lambda}}\hat{\phi} + i\sqrt{\frac{\lambda}{2}}\hat{n}$$

(g) Show that $[\hat{a}, \hat{a}^\dagger] = 1$.

In solving (h) and (i) below, you can directly quote the properties of $\hat{a}$, $\hat{a}^\dagger$, $\hat{a}^\dagger\hat{a}$, etc.

(h) Find the energy spectrum of $\hat{H}_2$.

(i) Compute the zero-point fluctuation of the number of Cooper pairs within the ground state of $\hat{H}_2$,

$$\langle(\Delta\hat{n})^2\rangle = \langle\hat{n}^2\rangle - \langle\hat{n}\rangle^2.$$

Next, let us try to understand transmons beyond the approximation $\hat{H}_2$. One has the freedom to work in the eigenbasis of $\hat{n}$, $\hat{n}|n\rangle = n|n\rangle$ with n an integer, or the eigenbasis of $\hat{\phi}$, $\hat{\phi}|\phi\rangle = \phi|\phi\rangle$ with $\phi \in [0, 2\pi]$. The basis transformation is facilitated by the equation

$$|n\rangle = \frac{1}{2\pi}\int_0^{2\pi} d\phi e^{-in\phi}|\phi\rangle$$

which looks awfully similar to a Fourier transform.

(j) Show that

$$e^{-i\hat{\phi}}|n\rangle = |n+1\rangle$$

and consequently

$$\hat{H} = \hbar\omega\left(\frac{\lambda}{2}\sum_{n=-\infty}^{+\infty} n^2|n\rangle\langle n| - \frac{1}{2\lambda}\sum_{n=-\infty}^{+\infty}\left(|n\rangle\langle n+1| + |n+1\rangle\langle n|\right)\right).$$