Qualifying Exam, Quantum Mechanics, August 2023

1. (40 points) At time $t = 0$, a particle undergoing one-dimensional simple harmonic oscillation is prepared in state $|\psi(t = 0)\rangle = (|0\rangle + e^{i\theta}|1\rangle)/\sqrt{2}$. Here $|n\rangle$ with $n = 0, 1, 2...$ are the eigenkets with energy $E_n = (n + 1/2)\hbar\omega$, θ is a given constant, and ω is the angular frequency of the oscillation.

   (a) If the position x of the particle is measured at $t = 0$, what is its expectation value?

   (b) Find the average kinetic energy of the particle at $t = 0$.

   (c) Find the state $|\psi(t)\rangle$ at later time $t > 0$.

   (d) Determine how the expectation value of the momentum p varies with time for $t > 0$.

   You may use the raising and lowering operators $a, a^\dagger$.

2. (40 points) Let $|l, m\rangle$ be the simultaneous eigenkets of angular momentum $\mathbf{L}^2$ and L_z with eigenvalues $l(l + 1)\hbar^2$ and $m\hbar$, respectively. The rotational states of a diatomic molecule are restricted to be superpositions of $|1, 1\rangle$, $|1, 0\rangle$, and $|1, -1\rangle$. (You do not need to worry about states with $l \neq 1$.)

   (a) Find the matrix form of operator L_x, the x-component of angular momentum, in basis $|1, 1\rangle$, $|1, 0\rangle$, and $|1, -1\rangle$. You may use operator $L_\pm$ and quote their properties directly from the textbook.

   (b) Suppose the molecule is in state $|1, 0\rangle$. If L_x is measured, what are the possible outcomes and the corresponding probabilities?

   (c) Compute the uncertainty of L_x in state $|1, 0\rangle$.

   (d) Suppose the molecule is instead in a mixed state described by density matrix

   $$\rho = \begin{pmatrix} 1/2 & 1/4 & 0 \\ 1/4 & 1/3 & 0 \\ 0 & 0 & 1/6 \end{pmatrix}$$

   in the same basis. Find the ensemble average $[L_z]$.

3. (20 points) Two electrons are trapped inside a cubic box of side length L. The walls of the box are impenetrable. Neglect the interaction between the electrons, but do remember electrons carry spin 1/2 and have mass m. For this problem, detailed calculations are not required, and you may quote directly the results for particle in infinite well.

   (a) What is the total energy, and the spin state, of the two electrons in the *ground state*? [7 points]

   (b) Is the *first excited state* degenerate? Why? [7 points]

   (c) Assume the system is initially in ground state. Then, suddenly, the walls are taken off so the electrons can now move freely. Describe qualitatively the time evolution of the "electron cloud" (the probability density of the electron) afterwards. You may sketch. [6 points]