Quantum Mechanics Qualifying Exam

Fall 2022

August 17 (9:00 am - 12:00 pm)

1. A particle is in the n^{th} energy state $\psi_n(x)$ of an infinite square well potential with width L. Determine the probability $P_n(1/a)$ that the particle is confined to the first $1/a$ fraction of the full width of the well (i.e. $0 < x < L/a$). For which states n is the expectation from classical mechanics reproduced?

2. An ensemble of particles arrives at the detector which measures their magnetic moment along some direction. Three different values of magnetic moment are found in these measurements (expressed in the instrument's units), and the fractions of all particles found with these magnetic moments are given in this table:

magnetic moment	-1	0	1
fraction of the incoming particles	20%	45%	35%

   (a) Magnetic moment is a property of particles which have non-trivial angular momentum. What angular momentum quantum numbers l and m do these particles have?

   (b) Is it possible to prepare this ensemble by making the same quantum measurement on a set of particles which are initially in the same quantum state, and then collecting only those particles which exhibited one particular measurement outcome? If the answer is "yes", then construct the density matrix for this scenario from the given information.

   (c) Is it possible to prepare this ensemble just by applying magnetic field B to a vapor of particles at some temperature T? If yes, what magnetic field and temperature would be needed? [Hint: in a thermal equilibrium, the probability of a state is proportional to $e^{-E/kT}$, where E is the state's energy and k is Boltzmann's constant]

   (d) If this is known to be a thermal ensemble, deduce that the particles must have experienced some anisotropy with their angular momentum, which goes beyond that created by the Zeeman effect in an externally applied magnetic field. In other words, show that the particles' Hamiltonian must have had the form

$$H = \text{const.} + aL_z + bL_z^2$$

   at the time their thermal ensemble was created, where both $a, b \neq 0$. (the constant term contains the kinetic energy L^2 of the particles, $a \propto B$ is the Zeeman term, and b is a residual anisotropy when $B = 0$). Determine the ratio b/a from the available data.

   (e) Why do we not need to consider a more general Hamiltonian $H = \text{const.} + aL_z + bL_z^2 + cL_z^3 + \cdots$ in this analysis? Explain.

3. A large number N of spin 1/2 particles are prepared in the same eigenstate of the operator S_x. This ensemble goes through a series of Stern-Gerlach-type of measurements as follows:

- The first measurement lets the $s_z = -\hbar/2$ particles pass, and blocks the $s_z = +\hbar/2$ particles.
- The second measurement blocks the $s_n = -\hbar/2$ particles and lets the $s_n = +\hbar/2$ ones pass. Here, s_n is the eigenvalue of the spin projection operator $\mathbf{S}\hat{\mathbf{n}}$ along the direction $\hat{\mathbf{n}}$, and the unit-vector $\hat{\mathbf{n}}$ makes an angle θ with the z-axis in the xz-plane.
- The final step measures S_x.

Calculate the numbers of particles N_{x+}, N_{x-} detected in the states $s_x = \hbar/2$ and $s_x = -\hbar/2$ respectively after the final step. In which quadrant should the θ value be to maximize N_+?

4. A box containing a particle is divided into a right and left compartment by a thin partition. If the particle is known to be on the right (left) side with certainty, its state is represented by a "position" eigenstate $|R\rangle$ ($|L\rangle$), where we have neglected spatial variations within each half of the box. The most general state vector can be written as:

$$|\alpha\rangle = |R\rangle\langle R|\alpha\rangle + |L\rangle\langle L|\alpha\rangle = C_R|R\rangle + C_L|L\rangle$$

where $C_R = \langle R|\alpha\rangle$ and $C_L = \langle L|\alpha\rangle$ can be regarded as "wavefunctions". The particle can tunnel through the partition. This tunneling effect is characterized by the Hamiltonian:

$$H = \Delta\Big(|L\rangle\langle R| + |R\rangle\langle L|\Big)$$

where Δ is a real number with units of energy.

(a) Find the normalized vectors of stationary states. What are the corresponding energies?

(b) Suppose the system is in some arbitrary initial state $|\alpha\rangle$ at $t = 0$, specified by $C_R = \langle R|\alpha\rangle$ and $C_L = \langle L|\alpha\rangle$ as above. What is the system's state at any time $t > 0$?

(c) Suppose that the particle is on the right side of the partition with certainty at $t = 0$. What is the probability of observing the particle on the left side as a function of time?

Useful formulas:

- Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$e^{i\boldsymbol{\sigma}\hat{\mathbf{n}}\theta} = \cos\theta + i(\boldsymbol{\sigma}\hat{\mathbf{n}})\sin\theta$$