Quantum Mechanics Qualifying Exam

Fall 2021

August 18 (9:00 am - 12:00 pm)

1. Consider a physical system with a three-dimensional state space. An orthonormal basis of the state space is chosen. In this basis, the Hamiltonian is represented by the matrix:

$$H = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

(a) What are the possible results when the energy of the system is measured?

(b) A particle is in the state $|\psi\rangle$ represented in this basis as

$$|\psi\rangle \sim \frac{1}{\sqrt{3}} \begin{pmatrix} i \\ -i \\ i \end{pmatrix}$$

Find the expectation values $\langle H \rangle$, $\langle H^2 \rangle$ and the uncertainty ΔH in this state.

2. A state for a simple harmonic oscillator of frequency ω starts evolving (at $t = 0$) from an arbitrary superposition of two number states

$$|\alpha, 0\rangle = \cos\theta\, |n\rangle + e^{i\phi} \sin\theta\, |l\rangle$$

where θ, ϕ are real and $n > l$.

(a) Derive the state vector at arbitrary time t.

(b) What is the energy expectation value at time t? Is it a periodic function of time? If yes, what is the period?

(c) Calculate the expectation value of potential energy at time t. Is it a periodic function of time? If yes, what is the period?

3. A molecule is rotating around its center of mass. The Hamiltonian is

$$H = \frac{L_x^2 + L_y^2}{2I_a} + \frac{L_z^2}{2I_b}$$

where I_a and I_b are the moments of inertia, and L_x, L_y, L_z are the orbital angular momentum operators.

(a). Find the energy eigenvalues and eigenstates.

(b) Consider a state described by the normalized angular wave function

$$\psi(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \sin\theta \cos\phi$$

where θ and ϕ the polar and azimuthal angles respectively. Compute the expectation value of L_z in the state ψ.

(c). Suppose $L_{\rm z}$ is measured in the above state ψ and the result is $+\hbar$. Immediately afterwards, L_x is measured. Find the uncertainty (standard deviation) ΔL_x in this measurement.

4. In a hydrogen atom, the wavefunction $\psi(\mathbf{r})$ describes the motion of the electron relative to the proton (i.e. $\mathbf{r} = \mathbf{r}_{\rm e} - \mathbf{r}_{\rm p}$, where $\mathbf{r}_{\rm e}$ and $\mathbf{r}_{\rm p}$ are the electron's and proton's coordinates respectively). If the atom is localized at the origin, show that the probability density of the proton is:

$$P_{\rm p}(\mathbf{r}_{\rm p}) = \left(\frac{m_{\rm e} + m_{\rm p}}{m_{\rm e}}\right)^3 \left|\psi\left(\frac{m_{\rm e} + m_{\rm p}}{m_{\rm e}} \mathbf{r}_{\rm p}\right)\right|^2$$

[Hint: Since you are implicitly given $\psi(\mathbf{r})$, you'll need to work with the center-of-mass $\mathbf{R}$ and relative $\mathbf{r}$ coordinates. Start by constructing the expression for $P_{\rm p}$ in terms of the atom's *two-body* wavefunction. Use the fact that the atom is localized at the origin, i.e. the two-body wavefunction pins $\mathbf{R}$ to zero.]