Quantum Mechanics Qualifying Exam

Fall 2018

August 23 (9:00 am - 12:00 pm), Exploratory Hall 1004

1. A rigid body with momentum of inertia I_z has a fixed axis of rotation and can rotate freely in the xy plane. Let ϕ be the angle that measures the amount of rotation

   (a) Find the energy eigenvalues and the corresponding normalized eigenfunctions of this rotating body at all times. (The classical energy of a rotating body is $L^2/2I$, where L is the angular momentum)

   (b) At time $t = 0$ the rotator is described by a wave-packet $\psi(\phi, 0) = A\sin^2\phi$. Find $\psi(\phi, t)$ for $t > 0$.

2. An electron with energy $E = 1$ eV is incident upon a rectangular barrier of potential energy $V_0 = 2$ eV. About how wide must the barrier be so that the transmission probability is 10^{-3}? Electron mass is $m = 9.1 \times 10^{-31}$ kg. [Hint: note the word "about". A quick sensible approximation is accepted for full credit. The exact calculation is feasible in an exam, but long and perilous - avoid at all costs.]

3. An electron with kinetic energy E is moving freely along the y-direction in a uniform magnetic field $\mathbf{B} = B\hat{\mathbf{y}}$. Its z-projection of spin is measured at $t = 0$ and found to be $+\hbar/2$.

   (a) Find the electron's complete spinor wavefunction:

   $$\psi(\mathbf{r}, t) = \begin{pmatrix} \psi_\uparrow(\mathbf{r}, t) \\ \psi_\downarrow(\mathbf{r}, t) \end{pmatrix}$$

   where $\psi_\uparrow \equiv \langle \uparrow | \psi \rangle$ and $\psi_\downarrow \equiv \langle \downarrow | \psi \rangle$ are the probability amplitudes of detecting the electron in the state $S^z \in \{\uparrow, \downarrow\}$ at position $\mathbf{r}$ and time t.

   (b) Calculate the expectation values for measurements of the observables S^x, S^y, S^z as functions of time.

   (c) How would the wavefunction change if the electron was initially set to move in the x-direction? Explain qualitatively, there is no need to solve the Schrodinger equation

4. A charged particle in a one-dimensional world is trapped by a harmonic oscillator potential:

   $$V(x) = \frac{kx^2}{2}$$

   The system is also placed in a uniform and constant external electric field E, which adds $\Delta H = -qEx$ to the Hamiltonian if the particle's charge is q.

   (a) Write the Hamiltonian in terms of the usual ladder operators a and $a^\dagger$.

   (b) Calculate the correction $\Delta\mathcal{E} = \Delta\mathcal{E}_1 + \Delta\mathcal{E}_2$ of the ground-state energy at the first and second order in the electric field strength E using the formulas known in perturbation theory:

   $$\Delta\mathcal{E}_1 = \langle 0 | \Delta H | 0 \rangle$$

$$\Delta\mathcal{E}_2 = \sum_{n=1}^{\infty} \frac{|\langle 0|\Delta H|n\rangle|^2}{\mathcal{E}_0 - \mathcal{E}_n}$$

where $|n\rangle$ and $\mathcal{E}_n$ are the Hamiltonian eigenstates and corresponding eigenvalues in the absence of electric field, $E = 0$.

(c) Find a "clever trick" to calculate the exact ground-state energy level for $E \neq 0$. Compare it with the result in part (b)

5. A molecule is made up of three identical atoms at the corners of an equilateral triangle. This molecule is converted to an ion by adding one electron to it, with some probability amplitude on each atomic site. Suppose the matrix element of the added electron's Hamiltonian on any two adjacent sites $i, j \in \{1, 2, 3\}$, $i \neq j$ is $\langle i|H|j\rangle = -a$.

   (a) Calculate the energy level splittings (differences).

   (b) Suppose the molecule is placed in the xy-plane and one of its triangle sides is aligned with the x-axis while the third "top" atom sits on the y-axis. How do the energy levels change if an electric field is applied in the y-direction so that the potential energy for the electron on the "top" atom is lowered by b? Assume $|b| \ll |a|$.