Classical Electrodynamics

Ph.D. Qualifying Examination

Monday, August 17, 2026 1:00 PM–4:00 PM

Duration: 180 minutes Total: 100 points

Candidate G#: ______________________________

Instructions

1. You may use one textbook and the attached formula sheet. Attempt all four problems.
2. No calculators, phones, computers, internet-connected devices, or other electronic aids are permitted.
3. Show all essential steps; unsupported final answers receive little or no credit.
4. State the boundary conditions used and the direction of every interface normal.
5. Earlier results may be quoted in later parts. If needed, state an assumed intermediate result clearly and continue.
6. Clearly define any additional notation you introduce.

Problem 1. Green's Function for a Grounded Sphere [25 points]

A grounded conducting spherical shell of radius a encloses the volume charge density

$$\rho(r,\theta) = \rho_0 \frac{r}{a}\cos\theta, \qquad 0 \le r < a.$$

The conductor is maintained at zero potential. The following Green's function is provided:

$$G(\mathbf{r},\mathbf{r}') = \sum_{\ell=0}^{\infty} \left[\frac{r_<^{\ell}}{r_>^{\ell+1}} - \frac{(rr')^{\ell}}{a^{2\ell+1}} \right] P_\ell(\cos\gamma).$$

(a) [**6 points**] Interpret the supplied Green's function. Define $r_<$, $r_>$, and γ; state the differential equation and boundary condition that it satisfies; and explain the roles of the two terms in square brackets. Verify directly that the boundary condition is satisfied on $r = a$.

(b) [**5 points**] Write the Green-function representation of the electrostatic potential, including the correct normalization. Use the angular dependence of the source and angular orthogonality to show that only the $\ell = 1$ term contributes, and reduce the solution to a one-dimensional radial integral.

(c) [**6 points**] Evaluate the radial integral and obtain $\Phi(r,\theta)$ everywhere inside the cavity. Verify explicitly that the result is regular at the origin and vanishes at $r = a$.

(d) [**3 points**] Find the induced surface charge density $\sigma_{\text{ind}}(\theta)$ on the *inner surface* of the conductor. Clearly state your normal convention.

(e) [**5 points**] Calculate the electric dipole moment of the volume charge and the electric dipole moment of the induced charge on the inner surface. Show that they cancel. Using the pure-$\ell = 1$ character established above, relate this cancellation to the vanishing exterior field of the grounded conductor.

Problem 2. A Non-Axisymmetric Electric Quadrupole [25 points]

Four point charges lie in the xy plane:

position	charge
$(+a, 0, 0),\ (-a, 0, 0)$	$+q$ at each point,
$(0, +a, 0),\ (0, -a, 0)$	$-q$ at each point.

Use the traceless Cartesian quadrupole tensor

$$Q_{ij} = \sum_n q_n \left(3x_{n,i}x_{n,j} - r_n^2 \delta_{ij}\right).$$

(a) **[3 points]** Show that the total charge and electric dipole moment vanish.

(b) **[7 points]** Calculate all nonzero components of Q_{ij}. Determine its eigenvalues and principal axes, and verify that the tensor is traceless as a consistency check.

(c) **[6 points]** Determine the leading far-field potential for $r \gg a$ and express it in spherical coordinates (r, θ, ϕ). Identify the azimuthal symmetry of the result.

(d) **[9 points]** The entire charge configuration is rotated rigidly through an angle α about the z axis and placed in the external potential

$$\Phi_{\text{ext}}(x, y, z) = -E_0 z + \frac{\Gamma}{2}(x^2 - y^2),$$

where Γ is a constant field-gradient parameter. Using either direct summation or the quadrupole interaction formula on the formula sheet, find the interaction energy $U(\alpha)$. Explain why the uniform-field term does not contribute, and determine and classify the stationary orientations for $q\Gamma > 0$. Orientations are understood modulo the π rotational symmetry of the charge configuration.

Problem 3. A Spherical Dielectric Shell in a Uniform Field [25 points]

A spherical vacuum cavity of radius a is surrounded by a concentric, homogeneous, linear dielectric shell of outer radius b and relative permittivity ϵ_r, so that the permittivity of the shell is $\epsilon = \epsilon_r\epsilon_0$. The region $r > b$ is vacuum. There is no free charge anywhere. Far from the shell,

$$\mathbf{E} \longrightarrow E_0\hat{\mathbf{z}}.$$

Define

$$\eta = \left(\frac{a}{b}\right)^3.$$

(a) **[3 points]** Using axial symmetry, regularity at the origin, and the far-field condition, justify the following $\ell = 1$ parameterization:

$$\begin{aligned}
\Phi_1 &= -E_0 A r\cos\theta, && 0 < r < a,\\
\Phi_2 &= -E_0\left(Br - C\frac{b^3}{r^2}\right)\cos\theta, && a < r < b,\\
\Phi_3 &= -E_0\left(r - F\frac{b^3}{r^2}\right)\cos\theta, && r > b.
\end{aligned}$$

(b) **[7 points]** Apply the electrostatic boundary conditions at $r = a$ and $r = b$. Derive four algebraic equations for A, B, C, F in terms of ϵ_r and η.

(c) **[9 points]** It is convenient to define

$$\Delta = (\epsilon_r + 2)(2\epsilon_r + 1) - 2\eta(\epsilon_r - 1)^2.$$

Solve for the two coefficients A and F that determine the observable fields outside the dielectric material. You need not report B and C unless they are useful in your derivation. Hence determine

(i) the uniform electric field in the central cavity, and

(ii) the induced electric dipole moment of the entire coated object.

(d) **[6 points]** Check the physical content of your answer in the following limits:

(i) $\epsilon_r = 1$ or $a \to b$;

(ii) $a \to 0$, for which the induced dipole moment must reduce to that of a solid dielectric sphere;

(iii) $\epsilon_r \to \infty$ with $a < b$ fixed.

Problem 4. A Permeable Cylinder in a Transverse Field [25 points]

An infinitely long cylinder of radius R and permeability μ is embedded in vacuum. Its axis is the z axis. There are no free currents anywhere. Far from the cylinder,

$$\mathbf{H} \longrightarrow H_0\hat{\mathbf{x}}.$$

Use cylindrical coordinates (r, ϕ, z), with $\phi = 0$ on the positive x axis.

(a) [**3 points**] Explain why a *single-valued* magnetic scalar potential may be introduced both inside and outside the cylinder, paying attention to the topology of the exterior region. Using symmetry, regularity, and the far-field condition, justify

$$\Phi_m^{\text{in}} = -AH_0 r\cos\phi, \qquad r < R,$$

$$\Phi_m^{\text{out}} = -H_0 r\cos\phi + CH_0\frac{R^2}{r}\cos\phi, \qquad r > R,$$

where $\mathbf{H} = -\nabla\Phi_m$.

(b) [**6 points**] Apply the magnetostatic boundary conditions at $r = R$ and determine A and C.

(c) [**5 points**] Find $\mathbf{H}$ and $\mathbf{B}$ inside the cylinder. Also give H_r and H_ϕ outside the cylinder.

(d) [**8 points**] Find the magnetization $\mathbf{M}$ inside the cylinder and the bound volume and surface current densities, $\mathbf{J}_b$ and $\mathbf{K}_b$. Show that the net z-directed bound current integrated around the circumference vanishes, and briefly interpret the signs of $K_{b,z}$ on the upper and lower halves of the cylinder.

(e) [**3 points**] Discuss the limits $\mu \to \infty$ and the formal limit $\mu \to 0^+$, the latter interpreted as an ideal perfect-diamagnet or flux-exclusion limit.

Formula Sheet

Electrostatics

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\gamma), \qquad \int P_\ell(\cos\theta) P_{\ell'}(\cos\theta)\, d\Omega = \frac{4\pi}{2\ell+1}\delta_{\ell\ell'}.$$

For the $\ell = 1$ angular factor,

$$P_1(\cos\gamma) = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi'), \qquad \int P_1(\cos\gamma)\cos\theta'\, d\Omega' = \frac{4\pi}{3}\cos\theta.$$

For a localized charge distribution,

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{r} + \frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} + \frac{1}{2r^3} Q_{ij}\hat{r}_i\hat{r}_j + \cdots\right],$$

$$q = \int \rho\, d^3r, \qquad \mathbf{p} = \int \rho\, \mathbf{r}\, d^3r, \qquad Q_{ij} = \int \rho(3x_ix_j - r^2\delta_{ij})\, d^3r.$$

For the traceless convention above, when Φ_{ext} obeys Laplace's equation near the source,

$$U_{\text{quad}} = \frac{1}{6} Q_{ij}\, \partial_i\partial_j \Phi_{\text{ext}}.$$

At an interface with normal $\hat{\mathbf{n}}$ directed from medium 1 to medium 2,

$$\hat{\mathbf{n}}\times(\mathbf{E}_2-\mathbf{E}_1) = 0, \qquad \hat{\mathbf{n}}\cdot(\mathbf{D}_2-\mathbf{D}_1) = \sigma_f.$$

For a linear dielectric, $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{P} = (\epsilon-\epsilon_0)\mathbf{E}$.

Magnetostatics

$$\nabla\times\mathbf{H} = \mathbf{J}_f, \qquad \nabla\cdot\mathbf{B} = 0, \qquad \mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M}).$$

If $\nabla\times\mathbf{H} = 0$, then $\mathbf{H} = -\nabla\Phi_m$ locally. A single-valued Φ_m exists throughout a region provided

$$\oint_C \mathbf{H}\cdot d\boldsymbol{\ell} = 0$$

for every closed contour C in that region. Simple connectivity is sufficient but not necessary.

At an interface,

$$\hat{\mathbf{n}}\times(\mathbf{H}_2-\mathbf{H}_1) = \mathbf{K}_f, \qquad \hat{\mathbf{n}}\cdot(\mathbf{B}_2-\mathbf{B}_1) = 0.$$

For a linear magnetic material,

$$\mathbf{B} = \mu\mathbf{H}, \qquad \mathbf{M} = \left(\frac{\mu}{\mu_0}-1\right)\mathbf{H}, \qquad \mathbf{J}_b = \nabla\times\mathbf{M}, \qquad \mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}}.$$

For a scalar f and vector $\mathbf{A}$ in cylindrical coordinates,

$$\nabla f = \hat{\mathbf{r}}\frac{\partial f}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial f}{\partial\phi} + \hat{\mathbf{z}}\frac{\partial f}{\partial z}, \qquad (\nabla\times\mathbf{A})_z = \frac{1}{r}\left[\frac{\partial(rA_\phi)}{\partial r} - \frac{\partial A_r}{\partial\phi}\right].$$