Classical Mechanics Qualifier (January 2026)
George Mason University

You are allowed to use your graduate textbook during the exam.
3 problems | Total 100 points

Problem 1 (20pts) A particle of mass m moves along the x-axis in a potential

$$U(x) = -ax^2 + bx^4,$$

where a and b are positive constants.

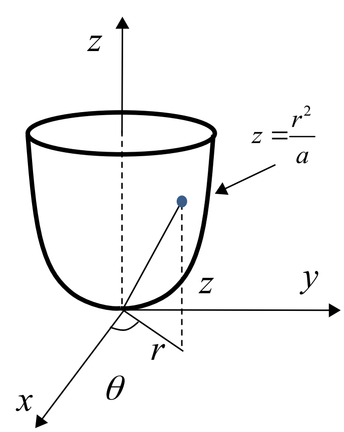
a) Write down the Hamiltonian describing this system and its Hamilton's equations of motion.

b) Find the equilibria of the system, and for each one determine if it is stable or unstable. Find the frequency of small oscillations ω around the stable points.

   [Hint: To calculate the frequency of small oscillations, just expand the resultant 2nd ODE around the equilibrium and pick out the frequency.]

c) State the Hamilton's Principle of Least Action. Consider a trajectory where the particle sits stationary at one of the stable equilibrium points for a total time T. What is the classical action S along this trajectory?

Problem 2 (40pts) A particle of mass m is constrained to move under the influence of gravity on the inside of a smooth parabolic surface of revolution given by $r^2 = az$. Use the Lagrange undetermined multiplier method to derive the constraint force for this problem. Write your answer as a vector in cylindrical coordinates.



Hint: You might want to use the two constants of motion E and l to simplify some of your expressions. The magnitude of the constraint force is proportional to

$$\left(1 + \frac{4r^2}{a^2}\right)^{-3/2}.$$

Problem 3 (40pts) Consider the orbits of a mass *m* in a central inverse-cube force,

$$F = \frac{-k}{r^3},$$

where *k* is a positive constant. Solve the radial equation of motion for *r* as a function of the angle variable θ on the orbital plane and classify them as bound or unbound for each of the following three cases:

i. Large angular momentum: $l > \sqrt{mk}$

ii. Small angular momentum: $l < \sqrt{mk}$

iii. $l = \sqrt{mk}$

(l is the constant generalized momentum corresponding to the generalized coordinate θ.)