Classical Mechanics Qualifier Exam (19 January 2021)

NAME: G-NUMBER:

Important instructions: In your solutions explain the details of your derivations. Present your solutions in a clean and clear way.

(1.) Consider a particle that describes a circular orbit under the influence of an attractive central force directed toward a point on the circle.

(a) Derive the Lagrangian L in polar coordinates with a radial distance r and azimuthal angle of θ and sketch the problem.

(b) Derive the Lagrangian equations of motion.

(c) Write down the canonical momentum for θ, the equation of motion in θ-direction, and the first integral involving the constant magnitude of the angular momentum, l.

(d) Derive a second order differential equation involving r (and the constant magnitude of the angular momentum) only.

(e) Derive the equation of the orbit

$$f(r) = \frac{l}{r^2}\left[\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{l}{mr^2}\frac{\mathrm{d}r}{\mathrm{d}\theta}\right) - \frac{l}{mr}\right] \tag{1}$$

(40 points)

(2.) A Hamiltonian of one degree of freedom has the form

$$H = \frac{p^2}{2a} - bqp\ \exp(-\alpha t) + \frac{ba}{2}q^2\exp(-\alpha t)[\alpha + b\exp(-\alpha t)] + \frac{kq^2}{2}, \tag{2}$$

where a, b, α, k are constants.

Find a Lagrangian corresponding to this Hamiltonian in terms of q and $\dot{q}$, eliminating p.

(20 points)

(3.) A point particle moves in space under the influence of a force derivable from a generalized potential U of the form:

$$U(\mathbf{r}, \mathbf{v}) = V(r) + \boldsymbol{\gamma}\cdot\mathbf{L}, \tag{3}$$

where $\mathbf{r}$ is the radius vector from a fixed point, $\mathbf{L}$ is the angular momentum about that point, and $\boldsymbol{\gamma}$ is a fixed vector in space.

(a) State the Lagrange's equation.

(b) Write down the equation for the generalized force Q_j as a function of the generalized potential $U(q, \dot{q})$?

(c) Find the components of the force on the particle in both Cartesian and spherical polar coordinates, on the the basis of the relationship between Q_j and $U(q, \dot{q})$ (the relationship from (b)).

(30 points)

(100 points in total.)