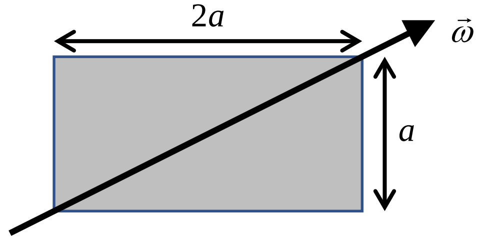
There are four questions, and each is worth 20 points.

1. Consider a uniform rectangular plate with mass M, side lengths a and $2a$, and negligible thickness. In the following, ignore gravitational and frictional forces.
   a. Identify the principal axes and derive the principal moments of inertia.



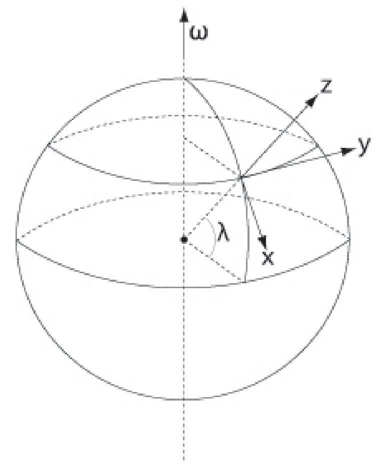
   b. The plate is made to rotate with a constant angular velocity around an axis coincident with a diagonal of the rectangle as shown. Express the angular momentum vector in terms of the body coordinate system consisting of the body's principal axes.
   c. What external torque is required to keep the plate rotating as described in part (b)?

2. Consider a vertical plane in a constant gravitational field. Let the origin of a coordinate system be located at some point in this plane. A particle of mass m moves in the vertical plane under the influence of gravity and an additional force $f = -Ar^{\alpha-1}$ directed towards the origin. Here, r is the distance from the origin; A and $\alpha\,(\neq 0\,\text{or}\,1)$ are constants. Ignore friction.
   a. Using polar coordinates in the vertical plane, find the Lagrangian and the Lagrangian equations of motion.
   b. Is the angular momentum about the origin conserved? If so, prove it. If not, show explicitly that the time rate of change of the angular momentum of the particle about the origin is equal to the moment of force (torque) about the origin.

3. If a particle is projected vertically upward to a height h above a point on the Earth's surface at a northern latitude λ (see the figure), show that it strikes the ground at a point

$$\left(\frac{4}{3}\sqrt{\frac{8h^3}{g}}\right)\omega\cos(\lambda)$$

towards the west, where ω is the angular speed of the earth's rotation. Neglect air resistance and only consider small heights h.



4. A system has a time-independent Hamiltonian $H_0(p,q)$ with p and q being the conjugate pair of canonical variables. The system is then perturbed periodically so that the Hamiltonian becomes

$$H = H_0(p,q) - \varepsilon q \sin(\omega t),$$

where ε and ω are the amplitude and angular frequency of the perturbation.

   a. Write the Hamilton equations of motion for the perturbed system.
   b. Given the generating function

$$F = F(q,P,t) - QP$$

   with

$$F_2(q,P,t) = qP - \left(\frac{\varepsilon q}{\omega}\right)\cos(\omega t),$$

   find the resulting canonical transformation.

   c. What is the new Hamiltonian $K(Q,P)$?
   d. Show that the Hamilton equations of motion in these new canonical variables remain in their standard form.